

ANALYTICAL FORMULATIONS TO THE METHOD OF VARIATION OF PARAMETERS IN TERMS OF UNIVERSAL Y'S FUNCTIONS

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SUMMARY: The method of variation of parameters still has a great interest and wide applications in mathematics, physics and astrodynamics. In this paper, universal functions (the Y's functions) based on Goodyear's time transformation formula were used to establish a variation of parameters method which is useful in slightly perturbed two-body initial value problem. Moreover due to its universality, the method avoids the switching among different conic orbits which are commonly occurring in space missions. The position and velocity vectors are written in terms of f and g series. The method is developed analytically and computationally. For the analytical developments, exact literal formulations for the differential system of variation of the epoch state vector are established. Symbolical series solution of the universal Kepler's equation was also established, and the literal analytical expressions of the coefficients of the series are listed in Horner form for efficient and stable evaluation. For computational developments of the method, an efficient algorithm was given using continued fraction theory. Finally, a short note on the method of solution was given just for the reader guidance.

Key words. celestial mechanics

1. INTRODUCTION

Variation of parameters method is well-known in the theory of differential equations. It is applied in celestial mechanics to a system of differential equations of the sixth order. Euler (1748) was the first to use the method in studying the mutual perturbations of Jupiter and Saturn. Lagrange (1808) developed the work of Euler and performed a series of papers

that posed the method of variation parameters in its final form. The main results of Lagrange's study was the system of planetary equations of orbital elements. In celestial mechanics, Lagrange extended the method of variation of parameters to the situation with velocity-dependent forces. (such treatment can be found, for example, in Brouwer and Clemence 1961, Efroimsky and Goldrieich 2003). Efroimsky and Goldrieich (2004) and Efroimsky (2005) implemented the variation of parameters method in terms

of the orbital elements defined in an accelerated frame.

For trajectories using low-level thrust, a simulation using variation of parameters is generally quite efficient. One well-known method is that described in (Bate et al. 1971), which uses Battin's universal variables and employs variation of the epoch state vector to describe the motion. It could be argued that the method of variation of parameters still an effective tool for solving differential equations, is highly popular among physicists, mathematicians and astronomers (Arakida and Fukushima 2001, Newman and Efroimsky 2003). Recently (Sharaf and Saad 2014, hereafter Paper I), a new set of universal functions; based on Goodyear's time transformation formula was developed analytically and computationally for a two body-initial value problem. These Y's functions are used here to develop a variation of parameters method which is useful in a slightly perturbed two-body initial value problem. Moreover, due to its universality, the method avoids the switching among the different conic orbits which are commonly occurring in space missions.

In the present approach, the position and velocity vectors are written in terms of f and g series. Lagrange coefficients are therefore expressed in terms of the Y's universal functions. The advantage of these functions is that they are convergent for values of the universal variable χ . Using the Lagrange coefficients to represent \vec{r} and \vec{v} , we can write the universal form of Kepler's equation. In the variation of parameter method we write the solution using \vec{r} and \vec{v} as the basis vectors and determine what values would be \vec{r}_0 and \vec{v}_0 . Performing some transformation rules as shown in Subsection 3.1, a new universal Kepler's equation is formulated. In the variation of parameters method \vec{r} replaces \vec{r}_0 and is treated as a constant and we get two equations for \vec{r}_0 and \vec{v}_0 . The last two equations could be integrated simultaneously with the first equation in Subsection 4.1. Given χ and α at a certain time t , the universal Y's functions could be evaluated by the algorithm given in Subsection 5.1.1.

The paper is organized as follows. In the next section, we review the universal Y's functions and their relation to the elementary functions in the two-body initial value problem. In Section 3, variation of the epoch state vector and transformation rules are discussed. Section 4 is devoted for the implementation of the variation of parameters problem. In Section 5, an efficient algorithm based on a continued fraction is established for the computational developments of the present method and evaluation of Y's functions. A symbolic series solution of the universal Kepler's equation is given in Section 6. Finally, we show, as a summary, the conclusions of this research.

2. THE UNIVERSAL Y'S FUNCTIONS AND THE TWO-BODY INITIAL VALUE PROBLEM

Goodyear's time transformation formula (Goodyear 1965) is given by:

$$\frac{dt}{d\chi} = r, \quad (1)$$

where χ is to be considered as a new independent variable - kind of generalized anomaly. In what follows, we develop some basic relations of these function due their rule in the analysis. The universal $Y_n(\chi; \alpha)$ functions are defined by:

$$Y_n(\chi; \alpha) = (\chi\sqrt{\mu})^n \sum_{k=0}^{\infty} (-1)^k \frac{(\alpha\mu\chi^2)^k}{(2k+n)!}, \quad (2)$$

clearly

$$Y_n(\chi; 0) = \frac{(\chi\sqrt{\mu})^n}{n!}, \quad (3)$$

α is just the inverse of the semi-major axis a given as:

$$\alpha = \frac{1}{a} = \frac{2}{r} - \frac{v^2}{\mu}, \quad (4)$$

and μ is the gravitational parameter. Eq. (4) is valid whatever the shape of the orbit is, namely, parabolic, elliptic or hyperbolic. Other useful properties of the Y's functions result directly from Eq. (2):

$$\frac{dY_n}{d\chi} = \sqrt{\mu}Y_{n-1}; n > 0, \quad (5)$$

$$\frac{dY_0}{d\chi} = -\alpha\sqrt{\mu}Y_1. \quad (6)$$

In the initial value problem of a two-body system, the position and velocity vectors \vec{r} and \vec{v} can be written in terms of the Lagrange coefficients f and g , and the basis vectors \vec{r}_0 and \vec{v}_0 :

$$\vec{r} = f\vec{r}_0 + g\vec{v}_0, \quad (7)$$

$$\vec{v} = \dot{f}\vec{r}_0 + \dot{g}\vec{v}_0, \quad (8)$$

where \vec{r}_0 and \vec{v}_0 are the initial position and velocity vectors, respectively. In terms of the Y's universal functions, the Lagrange coefficients are:

$$f = 1 - \frac{1}{r_0}Y_2(\chi; \alpha), \quad (9)$$

$$\dot{f} = -\frac{\sqrt{\mu}}{rr_0}Y_1(\chi; \alpha), \quad (10)$$

$$g = \frac{r_0}{\sqrt{\mu}}Y_1(\chi; \alpha) + \frac{\sigma_0}{\mu}Y_2(\chi; \alpha), \quad (11)$$

$$\dot{g} = 1 - \frac{1}{r}Y_2(\chi; \alpha), \quad (12)$$

where:

$$\sigma = \frac{1}{\sqrt{\mu}}\langle \vec{r}, \vec{v} \rangle, \quad (13)$$

and "0" subscripts indicate evaluation at the epoch time $t = t_0$ (the exception is Y_0 where subscript indicates the order of the universal function). Hereafter $\langle \vec{A}, \vec{B} \rangle$ will be used to denote the inner product of two vectors \vec{A} and \vec{B} . Using the Lagrange coefficients to represent \vec{r} and \vec{v} , we can write:

$$\sqrt{\mu}\sigma = \sigma_0 Y_0(\chi; \alpha) + (1 - r_0\alpha)\sqrt{\mu}Y_1(\chi; \alpha). \quad (14)$$

The universal form of Kepler's time equation becomes:

$$\begin{aligned} \Delta &= \sqrt{\mu}(t - t_0) = r_0 Y_1(\chi; \alpha) + \frac{\sigma_0}{\sqrt{\mu}} Y_2(\chi; \alpha) \\ &\quad + Y_3(\chi; \alpha). \end{aligned} \quad (15)$$

Finally, an important relation for χ is:

$$\mu\chi = \mu\alpha(t - t_0) + \sigma\sqrt{\mu} - \sigma_0. \quad (16)$$

3. VARIATION OF THE EPOCH STATE VECTOR

3.1. Transformation Rules

In the variation of parameters method (Burton and Melton 1992), we write the solution using \vec{r} and \vec{v} as basis vectors and then determine \vec{r}_0 and \vec{v}_0 . For the two-body system, to reach \vec{r} and \vec{v} at some particular time, we write:

$$\vec{r}_0 = F\vec{r} + G\vec{v}, \quad (17)$$

$$\vec{v}_0 = \dot{F}\vec{r} + \dot{G}\vec{v}, \quad (18)$$

and perform the transformations rules: $\chi \rightarrow -\chi$, $r \rightarrow r_0$, $\sigma \rightarrow \sigma_0$, in the original f and g terms, and $\Delta \rightarrow -\Delta = \sqrt{\mu}(t - t_0)$. The explanation of these transformation rules is due to the facts that movement from \vec{r} to \vec{r}_0 is equivalent to change χ to $-\chi$ and $\sqrt{\mu}(t - t_0)$ to $\sqrt{\mu}(t_0 - t)$. It should be noted that $Y_n(-\chi; \alpha) = (-1)^n Y_n(\chi; \alpha)$, so the first Lagrange coefficients become:

$$F = 1 - \frac{1}{r} Y_2(\chi; \alpha), \quad (19)$$

$$G = \frac{-r}{\sqrt{\mu}} Y_1(\chi; \alpha) + \frac{\sigma}{\mu} Y_2(\chi; \alpha), \quad (20)$$

while Eq. (14) and the universal Kepler's Eq. (15) are transformed, respectively, to:

$$\sqrt{\mu}\sigma_0 = \sigma Y_0(\chi; \alpha) - (1 - r\alpha)\sqrt{\mu}Y_1(\chi; \alpha), \quad (21)$$

$$\begin{aligned} \Delta &= \sqrt{\mu}(t - t_0) = r Y_1(\chi; \alpha) - \frac{\sigma}{\sqrt{\mu}} Y_2(\chi; \alpha) \\ &\quad + Y_3(\chi; \alpha). \end{aligned} \quad (22)$$

Finally, Eq. (16) is transformed to:

$$\mu\chi = \mu\alpha(t - t_0) - \sigma_0\sqrt{\mu} + \sigma. \quad (23)$$

Y_n are now functions of two variables χ and α , Eqs. (5) and (6) are therefore transformed to:

$$\frac{\partial Y_n(-\chi; \alpha)}{\partial(-\chi)} = \sqrt{\mu}Y_{n-1}(-\chi; \alpha), \quad (24)$$

then:

$$(-1)^n \frac{\partial Y_n(\chi; \alpha)}{\partial(\chi)} = -\sqrt{\mu}(-1)^{n-1} Y_{n-1}(\chi; \alpha), \quad (25)$$

$$\frac{\partial Y_n(\chi; \alpha)}{\partial\chi} = \sqrt{\mu}Y_{n-1}(\chi; \alpha), \quad (26)$$

$$\frac{\partial Y_0(\chi; \alpha)}{\partial\chi} = -\alpha\sqrt{\mu}Y_1(\chi; \alpha), \quad (27)$$

respectively.

3.2. Perturbed Motion

Having obtained the transformed equations, we have to consider the situation when a perturbation is introduced; In this case, we have to note that: (i) α is no longer constant, and so \dot{F} and \dot{G} must be computed by direct differentiation of F and G , taking into account the perturbation. (ii) The Lagrange coefficients are varying from the two-body form. Therefore, the perturbation causes no instantaneous changes in \vec{r} and the acceleration $\ddot{\vec{v}}$ results only from the perturbing forces and does not include changes due to the two-body reference motion.

4. IMPLEMENTATION OF VARIATION OF PARAMETERS PROBLEM

Since in the variation of parameters problem \vec{r} replaces \vec{r}_0 and is treated constant (Battin 1964, Bate et al. 1971), consequently:

$$\vec{r}_0 = \dot{F}\vec{r} + \dot{G}\vec{v} + G\ddot{\vec{v}}, \quad (28)$$

$$\vec{v}_0 = \ddot{F}\vec{r} + \ddot{G}\vec{v} + \dot{G}\ddot{\vec{v}}. \quad (29)$$

These equations are equivalent to the equations Eq. (6.26) in the book by Battin (1964).

4.1. Basic Evaluations

In order to evaluate the functions involved in Eqs. (28)-(29), some basic evaluations are needed first.

According to the notes mentioned in Subsection 3.2, we get from Eqs. (4), (13) and (23) that:

$$\frac{d\alpha}{dt} = -\frac{2}{\mu} \langle \vec{v}, \vec{v} \rangle, \quad (30)$$

$$\frac{d\sigma}{dt} = \frac{1}{\sqrt{\mu}} \langle \vec{r}, \vec{v} \rangle, \quad (31)$$

$$\frac{d\chi}{d\alpha} = (t - t_0) + \frac{1}{\mu^{3/2}} \langle \vec{r}, \vec{v} \rangle \frac{dt}{d\alpha}, \quad (32)$$

$$\frac{d\chi}{dt} = \alpha + \frac{1}{\mu^{3/2}} \langle \vec{r}, \vec{v} \rangle. \quad (33)$$

From Eq. (2) we can write:

$$\begin{aligned} 2 \frac{\partial Y_n}{\partial \alpha} &= (\chi \sqrt{\mu})^{n+2} \sum_{k=0}^{\infty} \frac{(-1)^k (\alpha \mu \chi^2)^k}{(n+2k+1)!} \cdot \\ \cdot \left\{ \frac{n}{n+2k+2} - 1 \right\} &= (\chi \sqrt{\mu})^{n+2} \sum_{k=0}^{\infty} (-1)^k (\alpha \mu \chi^2)^k \cdot \\ \cdot \left\{ \frac{n}{(n+2k+2)!} - \frac{1}{(n+2k+1)!} \right\}, \end{aligned} \quad (34)$$

that is:

$$\frac{\partial Y_n}{\partial \alpha} = \frac{1}{2} [n Y_{n+2} - \chi \sqrt{\mu} Y_{n+1}], \quad (35)$$

$$\frac{dY_n}{dt} = \left(\frac{\partial Y_n}{\partial \chi} \frac{d\chi}{d\alpha} \frac{d\alpha}{dt} + \frac{\partial Y_n}{\partial \alpha} \frac{d\alpha}{dt} \right). \quad (36)$$

Using Eq. (26), and Eqs. (30)-(35) into the above equation we deduce that:

$$\frac{dY_n}{dt} = Q_1^n \langle \vec{v}, \vec{v} \rangle + Q_2^n \langle \vec{r}, \vec{v} \rangle, \quad (37)$$

where

$$Q_1^n = \begin{cases} \frac{-2}{\mu} (\sqrt{\mu}(t-t_0)Y_{n-1} - \frac{1}{2}\chi\sqrt{\mu}Y_{n+1}) \\ + \frac{1}{2}nY_{n+2}; n \geq 1, \\ \frac{2}{\sqrt{\mu}} ((t-t_0)\alpha + \frac{1}{2}\chi) Y_1; n = 0, \end{cases} \quad (38)$$

$$Q_2^n = \begin{cases} \frac{1}{\mu} Y_{n-1}; n \geq 1 \\ -\frac{\alpha}{\mu} Y_1; n = 0. \end{cases} \quad (39)$$

Eqs. (30)-(35) and (37)-(39) are what we required to set up for the present subsection.

4.2. Evaluation of \dot{F} , \dot{G} , \ddot{F} and \ddot{G}

Differentiating Eqs. (19), (20) with respect to t , and then using Eqs. (33)-(35) we get:

$$\begin{aligned} \dot{F} &= \sum_{k=1}^4 C_k Y_k; \dot{G} = \sum_{k=0}^4 P_k Y_k, \\ \ddot{F} &= \sum_{k=0}^6 T_k Y_k; \ddot{G} = \sum_{k=0}^5 W_k Y_k, \end{aligned} \quad (40)$$

where C_k, P_k, T_k and W_k are given in Appendix.

5. EVALUATION OF THE Y'S FUNCTIONS

5.1. Gautschi's Algorithm for Continued Fraction Evaluation

In fact, continued fraction expansions are generally far more efficient tools for evaluating the classical functions than the more familiar infinite power series. Their convergence is typically faster and more extensive than in series. Due to the importance of accurate evaluations and the efficiency of continued fractions, we purpose to use them as the computational tools for evaluating the Y's functions. There are several methods available for evaluation is continued fraction. Traditionally, the fraction is either computed from the bottom up, or the numerator and denominator of the n^{th} convergent were accumulated separately with three-term recurrence formulae. The drawback of the first method is that obviously one has to decide how far down the fraction he goes to ensure convergence (i.e. before starting computations, we have to determine the number of iterations to ensure convergence). The drawback of the second method is that the numerator and denominator rapidly overflow numerically even though their ratio tends to a well defined limit. Thus, it is clear that an algorithm that works from top down, while avoiding numerical difficulties, would be ideal from a programming standpoint. Gautschi (1967) proposed a very concise algorithm to evaluate continued fraction from the top down and may be summarized as follows. If the continued fraction is written as:

$$\Omega = \frac{N_1}{D_1} \frac{N_2}{D_2 +} \frac{N_3}{D_3 + \dots}, \quad (41)$$

then initialize the following parameters

$$A_1 = 1, B_1 = N_1/D_1, \Omega_1 = N_1/D_1 \quad (42)$$

and iterate ($k = 1, 2, \dots$) according to:

$$A_{k+1} = \frac{1}{1 + \frac{N_{k+1}}{D_k D_{k+1}} A_k}, \quad (43)$$

$$B_{k+1} = (A_k - 1)B_k, \Omega_{k+1} = \Omega_k + B_{k+1}. \quad (44)$$

In the limit, the Ω sequence converges to the value of the continued fraction.

5.1.1 Continued Fraction Algorithm for Evaluating $Y_j(\chi; \alpha)$; $j = 0, 1, \dots, 6$

We shall consider evaluations of functions $Y_j(\chi; \alpha)$; $j = 0, 1, \dots, 6$, because these are the only functions that appear in the analysis (see Subsection 4.2). Battin(1999) succeeded to express his universal U's functions as continued fractions. We follow his methodology and developed the continued fractions representations of the Y's universal functions. Moreover we established the following algorithm for implementations of these representations on digital computers.

Computational Algorithm

Input: α, μ, χ

Output: $Y_j(\chi; \alpha)$; $j = 0, 1, 2, \dots, 6$

Computational Sequences

1- Compute a 's from:

$$a_j = -\frac{\alpha\mu\chi^2}{4(4j^2 - 1)}; j = 1, 2, \dots; a_0 = \frac{1}{2}\chi\sqrt{\mu}. \quad (45)$$

2- Compute u from the continued fraction:

$$u = \frac{a_0}{1+} \frac{a_1}{1+} \frac{a_2}{1+} \dots \quad (46)$$

by using Gautschi's algorithm of Subjection 5.1

$$3- A = 1 + \alpha\mu^2, \quad (47)$$

$$4- Y_0(\chi; \alpha) = (1 - \alpha u^2)/A, \quad (48)$$

$$5- Y_1(\chi; \alpha) = 2u/A, \quad (49)$$

$$6- Y_2(\chi; \alpha) = uY_1(\chi; \alpha), \quad (50)$$

$$7- q = \alpha u^2/A. \quad (51)$$

8- Compute γ_j ; $j = 1, 2, \dots$ from:

$$\gamma_0 = \frac{4}{3}Y_1^3(\chi; \alpha), \quad (52)$$

$$\gamma_n = \begin{cases} -\frac{(n+2)(n+5)}{(2n+1)(2n+3)}q; n \text{ odd} \\ -\frac{n(n-3)}{(2n+1)(2n+3)}q; n \text{ even} \end{cases}. \quad (53)$$

9- Compute $Y_3(2\chi; \alpha)$ from the continued fraction:

$$Y_3(2\chi; \alpha) = \frac{\gamma_0}{1+} \frac{\gamma_1}{1+} \frac{\gamma_2}{1+} \dots \quad (54)$$

by using Gautschi's algorithm of Subsection 5.1

$$10- Y_3(\chi; \alpha) = \frac{1}{2}Y_3(2\chi; \alpha) - Y_1(\chi; \alpha)Y_2(\chi; \alpha), \quad (55)$$

$$11- Y_4(\chi; \alpha) = \frac{1}{2}Y_3(\chi; \alpha) \{\sqrt{\mu}\chi + Y_1(\chi; \alpha)\} - \frac{1}{2}Y_2^2(\chi; \alpha). \quad (56)$$

12- Compute β_j ; $j = 0, 1, 2, \dots$ from

$$\beta_0 = -\frac{16}{15}Y_1^5(\chi; \alpha), \quad (57)$$

$$\beta_{2n+1} = -\frac{2(5+n)(5+2n)}{(5+4n)(7+2n)}q, \quad (58)$$

$$\beta_{2n} = -\frac{2n(2n-5)}{(3+4n)(5+4n)}q, \quad (59)$$

13- Compute B from the continued fraction

$$B = \frac{\beta_0}{1+} \frac{\beta_1}{1+} \frac{\beta_2}{1+} \dots \quad (60)$$

by using Gautschi's algorithm of Subjection 5.1

$$14- Y_5(2\chi; \alpha) = \frac{4}{3} \{Y_1^2(\chi; \alpha)Y_3(\chi; \alpha) + Y_1^3(\chi; \alpha) \cdot Y_2(\chi; \alpha) + \chi\sqrt{\mu}Y_2^2(\chi; \alpha)\} + \frac{8}{3}\chi\sqrt{\mu}Y_4(\chi; \alpha) + B, \quad (61)$$

$$15- Y_5(\chi; \alpha) = \frac{1}{2}Y_5(2\chi; \alpha) - \chi\sqrt{\mu}Y_4(\chi; \alpha) - Y_2(\chi; \alpha)Y_3(\chi; \alpha), \quad (62)$$

$$16- Y_6(\chi; \alpha) = \frac{1}{2} (Y_3^3(\chi; \alpha) - Y_2(\chi; \alpha)Y_4(\chi; \alpha)) - \frac{1}{4}\mu\chi^3Y_4(\chi; \alpha) + \sqrt{\mu}\chi Y_5(\chi; \alpha). \quad (63)$$

17- End

6. SYMBOLIC SOLUTION OF THE UNIVERSAL KEPLER'S EQUATION

In what follows we shall established symbolic solution of the universal Kepler's equation for two general epochs t_s and t_ℓ , so Kepler's equation becomes:

$$\sqrt{\mu}(t_\ell - t_s) = r_s Y_1(\chi_{\ell,s}; \alpha_s) + \frac{\sigma_s}{\sqrt{\mu}} Y_2(\chi_{\ell,s}; \alpha_s) + Y_3(\chi_{\ell,s}; \alpha_s), \quad (64)$$

where

$$\sqrt{\mu}(t_\ell - t_s) = \Delta_{\ell,s}. \quad (65)$$

Reversing series (64) leads to a solution for $\chi_{\ell,s}$ as:

$$\chi_{\ell,s} = \sum_{k=1}^N L_k \Delta_{\ell,s}^k, \quad (66)$$

where the infinite series is truncated to N terms. The polynomial was arranged in the Horner form. This is useful for an efficient and stable numerical evaluation. Assume that χ^n can be calculated using only $\log_2 n$ multiplications for integer n (Knuth 1981). For a polynomial of degree n the Horner form requires n multiplications and n additions. The expanded form, however, requires $\sum_{i=1}^n \log_2 i = \log_2 \Gamma(n+1)$ multiplications, which is already more than twice as expensive as for a polynomial of degree 10. Thus, one advantage of the Horner form is that the work involved in exponentiation is distributed across addition and multiplication which results in savings of some basic arithmetic operations. Another advantage is that the Horner form is more stable in numerical evaluations when compared with the expanded form. The reason for this is that each sum and product involves quantities which vary on a more evenly distributed scale. Because of space limitations, only the first nine coefficients are listed in Table I of Appendix A (Paper I).

7. OUTLINE OF METHOD OF SOLUTION

At the end it is worth to mention just the outline of the method of solution. Eqs. (28)-(29) could be integrated simultaneously with Eq. (30). The position and velocity vectors \vec{r} and \vec{v} are evaluated from the original Lagrange coefficients Eqs. (17)-(18). The universal variable χ at a given time, can be obtained from Eq. (15), either by numerical iteration, or via analytic series as in Section 6. Although the latter method is frequently used to obtain the first approximation (Sharaf and Sharaf 1998) for numerical iteration, it could be used to get a highly accurate value of χ by returning t_s to the formulation and periodically resetting the epoch during the integration (Burton and Melton 1992). This would allow Δ to remain small and the series representation of Eq. (64) with the nine coefficients of Appendix A (Paper I) are more than sufficient to get quite accurate values of χ . With the values of χ and α at a given time, the universal Y function could be evaluated by the continued fractions algorithm of Subsection 5.1.1. Full numerical applications of the formulations of the present paper will constitute a task to which we shall consider in a separate paper.

In concluding the present paper, a variation of parameters method is established analytically and

computationally. For analytical developments, exact formulations for the differential system of variation of the epoch state vector are established. A symbolical series solution of the universal Kepler's equation was also established, and analytical expressions for the coefficients of the series are listed in the Horner form for an efficient and stable evaluation. For computational developments of the method, an efficient algorithm using continued fraction theory was given.

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Appendix A: THE FORMULATIONS OF C_k, P_k, T_k AND W_k OF EQUATIONS (40)

$$C_1 = \frac{1}{r\mu}(2\Delta\langle\vec{v}, \dot{\vec{v}}\rangle - \langle\vec{r}, \dot{\vec{v}}\rangle), C_2 = 0,$$

$$C_3 = -\frac{\chi}{r\sqrt{\mu}}\langle\vec{v}, \dot{\vec{v}}\rangle, C_4 = \frac{2}{r\mu}\langle\vec{v}, \dot{\vec{v}}\rangle, \quad (67)$$

$$P_0 = \frac{r}{\mu^{3/2}}(2\Delta\langle\vec{v}, \dot{\vec{v}}\rangle - \langle\vec{r}, \dot{\vec{v}}\rangle),$$

$$P_1 = \frac{-\sigma}{\mu^2}(2\Delta\langle\vec{v}, \dot{\vec{v}}\rangle - \langle\vec{r}, \dot{\vec{v}}\rangle),$$

$$P_2 = -\frac{r\chi}{\mu}\langle\vec{v}, \dot{\vec{v}}\rangle, P_3 = \frac{(r + \sigma\chi)}{\mu^{3/2}}\langle\vec{v}, \dot{\vec{v}}\rangle,$$

$$P_4 = -\frac{2\sigma}{\mu^2}\langle\vec{v}, \dot{\vec{v}}\rangle, \quad (68)$$

$$T_0 = -\frac{1}{r\mu^2}((2\Delta\langle\vec{v}, \dot{\vec{v}}\rangle)^2 + \langle\vec{r}, \dot{\vec{v}}\rangle^2),$$

$$T_1 = \frac{1}{r\mu}(2\Delta(\langle\vec{v}, \ddot{\vec{v}}\rangle + \dot{\vec{v}}^2) + 2\sqrt{\mu}\langle\vec{v}, \dot{\vec{v}}\rangle - \langle\vec{v}, \ddot{\vec{v}}\rangle),$$

$$T_2 = -\frac{4\Delta\chi}{r\mu^{3/2}}\langle\vec{v}, \dot{\vec{v}}\rangle^2,$$

$$T_3 = -\frac{1}{r\mu^2}\left(\mu^{3/2}\chi(\langle\vec{v}, \ddot{\vec{v}}\rangle + \dot{\vec{v}}^2) + \langle\vec{v}, \dot{\vec{v}}\rangle(\alpha\mu^{3/2} + 6\Delta\langle\vec{v}, \dot{\vec{v}}\rangle - 2\langle\vec{r}, \dot{\vec{v}}\rangle)\right),$$

$$T_4 = \frac{1}{r\mu}(2(\Delta\langle\vec{v}, \ddot{\vec{v}}\rangle + \dot{\vec{v}}^2) - \chi^2\langle\vec{v}, \dot{\vec{v}}\rangle^2),$$

$$T_5 = \frac{5\chi}{r\mu^{3/2}}\langle\vec{v}, \dot{\vec{v}}\rangle^2, T_6 = -\frac{8}{r\mu^2}\langle\vec{v}, \dot{\vec{v}}\rangle^2, \quad (69)$$

$$W_0 = \frac{1}{\mu^{7/2}}\left(\sqrt{\mu}(2\Delta(2\Delta - 1)\sigma\langle\vec{v}, \dot{\vec{v}}\rangle + (\sigma - 2\sqrt{\mu} - 2\Delta\sigma)\langle\vec{r}, \dot{\vec{v}}\rangle) + r(2\mu^{5/2} - \langle\vec{r}, \ddot{\vec{v}}\rangle) + 2r\Delta(\langle\vec{v}, \ddot{\vec{v}}\rangle + \dot{\vec{v}}^2)\right),$$

$$W_1 = \frac{1}{\mu^4}(\mu(2r\alpha\Delta(2\Delta - 1)\sqrt{\mu} + r(4\Delta - 1)\mu\chi - 2\sigma(2\alpha\Delta + \sqrt{\mu}\chi))\langle\vec{v}, \dot{\vec{v}}\rangle + \mu(-1 + r\alpha)(-1 + 2\Delta - 1)\sqrt{\mu} + 2\alpha\sigma - r\mu\chi) \cdot \langle\vec{r}, \dot{\vec{v}}\rangle + \sigma(\langle\vec{r}, \ddot{\vec{v}}\rangle - 2(\mu^{5/2} + \Delta(\langle\vec{v}, \ddot{\vec{v}}\rangle + \dot{\vec{v}}^2)))$$

$$W_2 = \frac{1}{\mu^3}(-r\alpha\mu^2 + \sqrt{\mu}((\sigma\chi - 2\Delta(r + 2\sigma\chi)) \cdot \langle\vec{v}, \dot{\vec{v}}\rangle + \sigma\chi\langle\vec{r}, \dot{\vec{v}}\rangle) - r\chi(\langle\vec{v}, \ddot{\vec{v}}\rangle + \dot{\vec{v}}^2)),$$

$$W_3 = \frac{1}{\mu^{7/2}}(\alpha\mu^2\sigma + \sqrt{\mu}(-(\sigma - 2\Delta\sigma + r\mu^{3/2}\chi^2))\langle\vec{v}, \dot{\vec{v}}\rangle + (\sigma + \mu\chi)\langle\vec{r}, \dot{\vec{v}}\rangle + (r + \sigma\chi)(\langle\vec{v}, \ddot{\vec{v}}\rangle + \dot{\vec{v}}^2)),$$

$$W_4 = \frac{1}{\mu^4}(\mu^2\chi(3r + \varrho\chi)\langle\vec{v}, \dot{\vec{v}}\rangle - 2\sigma(\langle\vec{v}, \ddot{\vec{v}}\rangle + \dot{\vec{v}}^2)),$$

$$W_5 = -\frac{3}{\mu^{5/2}}(r + \sigma\chi)\langle\vec{v}, \dot{\vec{v}}\rangle. \quad (70)$$

**АНАЛИТИЧКА ФОРМУЛАЦИЈА МЕТОДА ВАРИЈАЦИЈЕ
ПАРАМЕТАРА У ОБЛИКУ УНИВЕРЗАЛНИХ Y ФУНКЦИЈА**

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Оригинални научни рад

Метод варијације параметара и даље је актуелан и користи се у многим областима математике, физике и астродинамике. У овом раду коришћене су универзалне (тзв. Y) функције које се заснивају на Гудјеровој формули за трансформацију времена како би дефинисали метод варијације параметара користан код проблема почетних услова у благо поремећеном систему два тела. Због своје универзалности, метод не захтева преласке на различите типове орбита, тј. конусне пресеке, до којих често долази код свемирских мисија. Вектори положаја и брзине написани су у об-

лику f и g редова. Метод је развијен аналитички и рачунски. У аналитичком извођењу добијена је тачна формулација система варијационих једначина вектора стања за одређену епоху. Добијено је и решење опште Кеплерове једначине у облику симболичког реда, као и аналитички изрази за коефицијенте реда у Хорнеровом облику, за ефикасно и стабилно израчунавање. За потребе рачунског развоја метода, дат је ефикасан алгоритам уз помоћ теорије верижних разломака. На крају, дат је и кратак осврт на метод решења, као смерница заинтересованом читаоцу.