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Invited review

HYPERBOLIC NORMAL FORMS AND INVARIANT MANIFOLDS. ASTRONOMICAL APPLICATIONS

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SUMMARY: In recent years, the study of the dynamics induced by the invariant manifolds of unstable periodic orbits in nonlinear Hamiltonian dynamical systems has led to a number of applications in celestial mechanics and dynamical astronomy. Two applications of main current interest are i) space manifold dynamics, i.e. the use of the manifolds in space mission design, and, in a quite different context, ii) the study of spiral structure in galaxies. At present, most approaches to the computation of orbits associated with manifold dynamics (i.e. periodic or asymptotic orbits) rely either on the use of the so-called Poincaré - Lindstedt method, or on purely numerical methods. In the present article we briefly review an analytic method of computation of invariant manifolds, first introduced by Moser (1958), and developed in the canonical framework by Giorgilli (2001). We use a simple example to demonstrate how hyperbolic normal form computations can be performed, and we refer to the analytic continuation method of Ozorio de Almeida and co-workers, by which we can considerably extend the initial domain of convergence of Moser's normal form.

Key words. celestial mechanics – methods: analytical

1. INTRODUCTION

The dynamical features of the *invariant manifolds* of unstable periodic orbits in nonlinear Hamiltonian dynamical systems is a subject that has attracted much interest in recent years, due to a number of possible applications in various problems encountered in the framework of celestial mechanics and dynamical astronomy.

The possibility to exploit the invariant manifolds of unstable periodic orbits in the neighborhood of the collinear libration points of the Earth - Moon, or the Earth - Sun system, in order to design low cost space missions, constitutes a new branch called *space manifold dynamics*. The reader is deferred to Perozzi and Ferraz-Mello (2010), and in particular to the review by Belló et al. (2010) in the same volume, or to Gómez and Barrabes (2011), for detailed reviews and a comprehensive list of references.

In a quite different context, the invariant manifolds of unstable periodic orbits in the co-rotation region of barred galaxies have been proposed as providing a mechanism for the generation and/or maintenance of *spiral structure* beyond co-rotation (Voglis et al. 2006, Romero-Gomez et al. 2006, 2007, Tsoutsis et al. 2008, 2009). Fig. 1 (Tsoutsis et al. 2008) shows an example of this mechanism. This figure shows the superposition of the unstable invariant manifolds of seven different unstable periodic orbits covering a domain from about 0.8 to twice the co-rotation radius in an N-body model of a barred-spiral galaxy. It is a basic fact that the unstable manifolds of one periodic orbit cannot have intersections either with themselves or with the unstable manifolds of

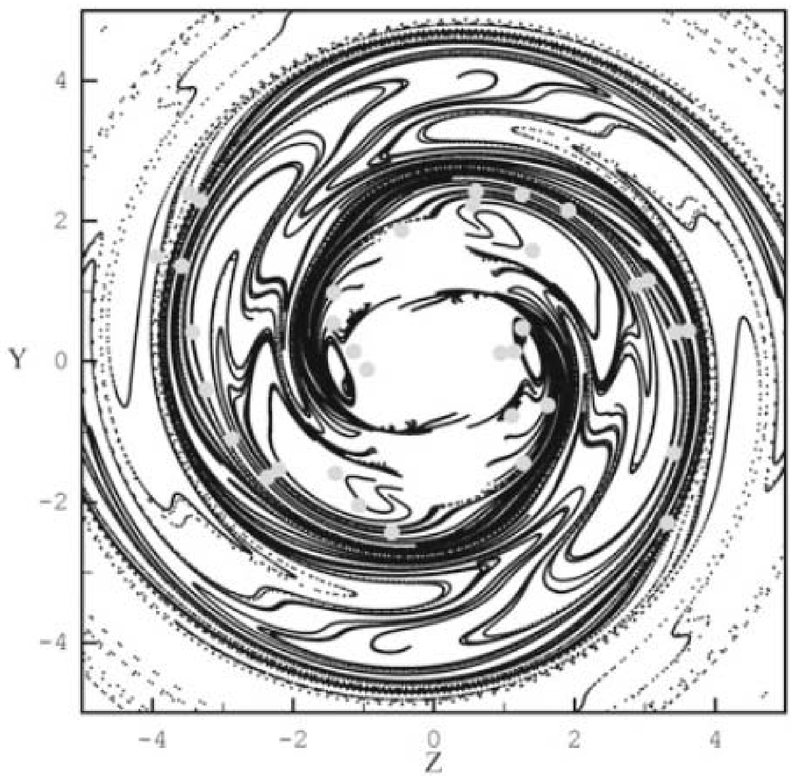

Fig. 1. *The projection in configuration space of the unstable invariant manifolds of seven different unstable periodic orbits in the co-rotation region of a N-body model of a barred-spiral galaxy (see Tsoutsis et al. 2008 for details).*

any other periodic orbit of equal energy. Due to this property, the manifolds of different periodic orbits develop in nearly parallel directions in the phase space, and their lobes penetrate one into the other, forming a pattern called the 'coalescence' of invariant manifolds (Tsoutsis et al. 2008). We then find that the latter has the characteristic shape of a bisymmetric set of spiral arms.

Viewed from a dynamical systems point of view, the invariant manifolds provide an underlying structure in a connected chaotic domain, which influences the laws by which the chaotic orbits evolve. In particular, the manifolds play a key role in characterizing the phenomenon of *chaotic recurrences*. The dynamical consequences induced by the geometric structure of the invariant manifolds are emphasized already in the work of H. Poincaré (1892). However, starting with Contopoulos and Polymilis (1993), an investigation of the manifolds' lobe dynamics and recurrence laws has been a subject of only relatively recent studies (see Contopoulos 2002 for a review).

The computation of the invariant manifolds in concrete dynamical systems can be realized by analytical or numerical methods, or by their combination.

In space manifold dynamics, we are often interested in computing simply unstable periodic orbits around the collinear libration points in the framework of the circular restricted three body problem, where, depending on the application, the primary and secondary bodies can be taken either as the Earth and the Moon, or the Sun and the barycenter of the Earth - Moon system. Of particular interest are the short period orbits lying in the plane (called 'horizontal Lyapunov orbit') and perpendicular to the plane (vertical Lyapunov orbit) of motion of the primary and secondary bodies, as well as the 1:1 resonant short period orbit called 'halo orbit'. A usual computational approach is to employ the Poincaré - Lindstedt method in order to compute the periodic orbits themselves in the form of a Fourier series (see Belló et al. 2010, Section 3). Then, exploiting the fact that the invariant manifolds of these orbits are tangent to the invariant manifolds of the linearized flow in the neighborhood of the periodic orbits, we can compute initial conditions along either the unstable or the stable manifold, whose numerical integration (forward or backward in time) produces asymptotic orbits lying on the unstable or stable invariant manifold, respectively. The accuracy of this method depends on i) the accuracy of approximation of the periodic orbits by Lindstedt series, and (ii) the accuracy of the numerical orbit integrator.

In the sequel, we will present a method of computation of the unstable periodic orbits and of their manifolds, due to Moser (1958, see also Siegel and Moser 1991). This is called the method of *hyperbolic normal form*. In its original form this method refers to a direct computation of the form of the phase space flow around unstable *equilibrium points* of Hamiltonian dynamical systems. This is achieved by introducing an appropriate transformation of the phase-space variables, such that the form of the invariant manifolds is trivial in the new variables. However, we will show below that no fundamental difficulty exists in passing from the study of unstable equilibria to the study of unstable periodic orbits using essentially the same method, provided that the periodic orbit of interest arises as a continuation of some unstable equilibrium point.

Moser's way of introducing transformations of variables does not guarantee the preservation of the canonical character of the flow in the new variables. However, a canonical form of the same theory using Lie generating functions was developed by Giorgilli (2001).

An important feature of both Moser's and Giorgilli's methods is the fact that the so-resulting normal forms have a finite domain of *convergence*. This sounds peculiar at first, since the resulting series are supposed to provide an analytic representation of *chaotic* orbits, while, on the other hand, it is well known that the Birkhoff series representing regular motions around elliptic equilibria are not convergent but only asymptotic. However, one can observe that the convergence of the hyperbolic normal form is due to the fact that the associated series contain no small divisors. In fact, in the Birkhoff series we have divisors of the form $m_1\omega_1 + m_2\omega_2$, with m_1, m_2 integers and ω_1, ω_2 real. But the construction of the hyperbolic normal form can be thought of as analogous to the construction of a Birkhoff's normal form in which we consider one of the two frequencies, say ω_2, to be *imaginary*, i.e. of the form, $\omega_2 = i\nu$, where ν is a real number. This number represents the absolute value of the (also real) logarithm of either of the eigenvalues of the monodromy matrix of the unstable periodic orbit generating the manifolds (see below). Thus, in the hyperbolic normal form the divisors are of the form $m_1\omega_1 + im_2\nu$, whereby it follows that a divisor's modulus can never become smaller than the minimum of $|\omega_1|$ and $|\nu|$.

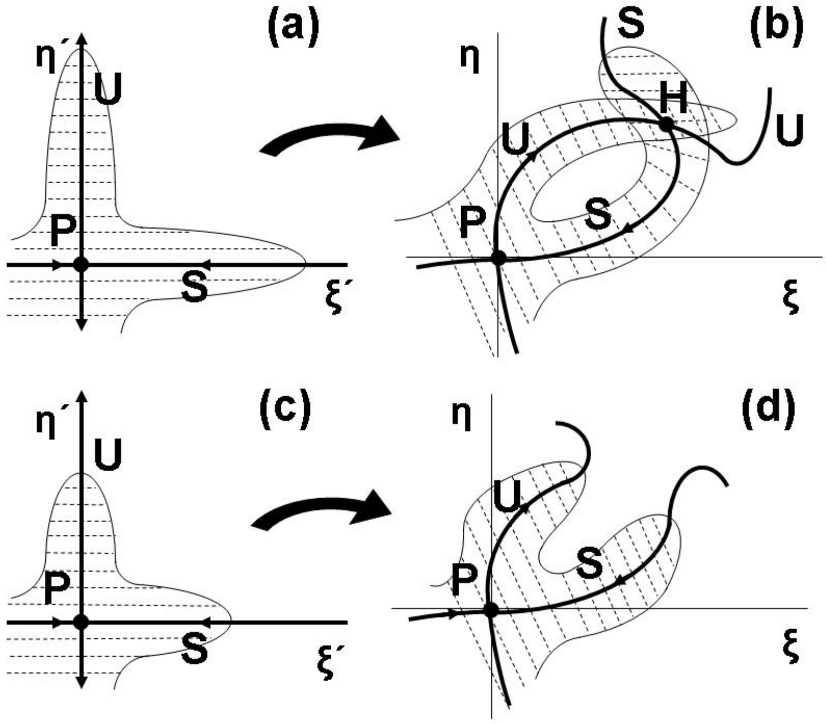

Fig. 2. *A schematic example of the transformation of the convergence domain of the hyperbolic normal form when passing from new to old canonical variables $(\xi', \eta') \rightarrow (\xi, \eta)$ (see text). (a) The shaded area represents the convergence domain around the unstable periodic orbit P, including a segment of the unstable (U) and stable (S) invariant manifolds of P, which, in these variables, coincide with the axes. (b) When passing to the old variables (ξ, η), the domain of convergence is transformed so that it includes a homoclinic point H. (c) Same as in (a) but for a smaller domain of convergence. Now, the image in old variables (d) contains no homoclinic point.*

This fact notwithstanding, the domain of convergence of the hyperbolic normal form is *finite*. Fig. 2 shows schematically the implications of this latter fact in the computation of the so-called homoclinic points, i.e. points where the stable and unstable manifolds of the same periodic orbit intersect. As made clear in Section 2 below, in a set of new canonical variables, say ξ', η' which are defined after the end of the normal form computation, the invariant manifolds correspond to the axes $\xi' = 0$ and $\eta' = 0$. The images of these axes in the corresponding original canonical variables ξ, η are tilted curves. On the other hand, the domain of convergence of the hyperbolic normal form in the (ξ', η') plane has the form of a shaded area, as in Figs. 2a and 2c. These figures represent two distinct cases regarding the size of the domain of convergence. Fig. 2a represents a case in which, when mapping the shaded area to a corresponding shaded area in the original canonical variables (ξ, η) (Fig. 2b), the segment of the invariant manifolds contained within the shaded area is long enough so as to include the first homoclinic intersection of the stable and unstable manifolds. When this happens, the hyperbolic normal form can be used to compute analytically the position of the corresponding homoclinic point. On the other hand, if the domain of convergence is small (shaded area in Fig. 2c), then its image in the old variables (Fig. 2d) does not contain a homoclinic point.

The question of how to predict whether or not the domain of convergence of a hyperbolic normal form contains one or more homoclinic points is open. In fact, there is only a limited number of studies of the numerical outcome of hyperbolic normal form computations in general. In this respect, an important work was done in the 90's by Ozorio de Almeida and collaborators (Da Silva Ritter et al. 1987, Ozorio de Almeida 1988, Ozorio de Almeida and Viera 1996, Viera and Ozorio de Almeida 1997), who actually proposed an extension of the method of Moser resulting in a considerable increase of the domain of convergence. We will examine this extension by a concrete example below. However, we mention that the implementation of even the original method in symplectic mappings rather than flows (Moser 1956) has given impressive results, as for example in Da Silva Ritter et al. (1987), where not only the first homoclinic point but also some oscillations of the invariant manifolds were possible to compute analytically (see Fig. 5 of Da Silva Ritter et al. (1987)).

The computations of Ozorio de Almeida and collaborators use the original version of Moser's normal form, which makes no use of generating functions or the canonical formalism. In the sequel, we present a simple application in a perturbed pendulum model using the canonical formalism instead, as proposed by Giorgilli (2001). We then give a concrete example of computation of the hyperbolic normal form, and also implement the extension proposed by Ozorio de Almeida within the same context. The example is presented in sufficient detail so as to provide i) a full explanation of the method, and ii) a numerical probe of its performance. However, we should stress that this subject is relatively new as far as concrete applications are concerned, and further study is required in order to establish the limits and usefulness of the method of hyperbolic normal forms.

2. NUMERICAL EXAMPLE

In order to give a concrete numerical example of computation of the hyperbolic normal form, we consider a periodically driven pendulum model given by the Hamiltonian:

$$H = \frac{p^2}{2} - \omega_0^2(1 + \epsilon(1+p)\cos\omega t)\cos\psi \ . \qquad (1)$$

A model of a form similar to Eq. (1) often appears in cases of resonances in astronomical systems. Introducing a dummy action I and its conjugate angle $\phi = \omega t$ we can write equivalently the Hamiltonian as:

$$H'(\psi, \phi, p, I) = \frac{p^2}{2} + \omega I - \omega_0^2(1+\epsilon(1+p)\cos\phi)\cos\psi \ . \qquad (2)$$

Fig. 3a shows the phase portrait for a rather high value of the perturbation ϵ, namely $\epsilon = 1$, when $\omega_0 = 0.2\sqrt{2}$, $\omega = 1$. The phase portrait is obtained by a stroboscopic plot of all points $(\psi(nT), p(nT))$ along particular orbits at the successive times $t = nT$, $n = 1, 2, ...$, where $T = 2\pi/\omega$ is the perturber's period. We observe that most trajectories are chaotic in the considered domain. In fact, only a small part of the libration domain, as well as two conspicuous 1:1 resonant islands and some other smaller islands host quasi-periodic trajectories.

The most important source of chaos in Fig. 3a is an unstable periodic orbit, called hereafter the orbit P, which is the continuation for $\epsilon \neq 0$ of the hyperbolic equilibrium point which exists for $\epsilon = 0$ at $\psi = \pi$, $p = 0$. This orbit generates the stable (W_s^P) and unstable (W_u^P) manifolds whose intersections with the surface of section correspond to the curves denoted W_s^P and W_u^P in Fig. 3b.

We now give the following definitions:

Let P be a periodic orbit of period T, and:

$$\mathcal{D}_P = \left\{ \Big(\psi_P(t), \phi_P(t), I_{\psi,P}(t), I_P(t)\Big), 0 \leq t \leq T \right\}$$

be the set of all points of the periodic orbit P parametrized by the time t. Let $\mathbf{q} = (\psi, \phi, I_\psi, I)$ be a randomly chosen point in the phase space. The minimum distance of the point $\mathbf{q}$ from the periodic orbit is defined as:

$$d(\mathbf{q}, P) = \min\{dist(\mathbf{q}, \mathbf{q}_p) \text{ for all } \mathbf{q}_P \in \mathcal{D}_p\}$$

where $dist()$ denotes the Euclidean distance. Finally, let $\mathbf{q}(t; \mathbf{q}_0)$ denote the orbit resulting from a particular initial condition $\mathbf{q}_0$, at $t = 0$.

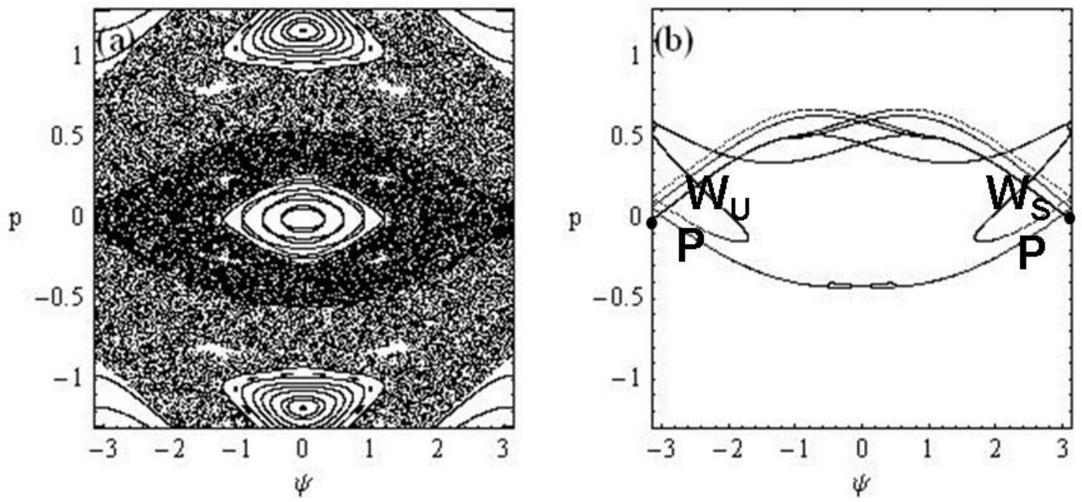

Fig. 3. *(a) Surfaces of section of the perturbed pendulum model (Hamiltonian (2)) for* $\epsilon = 1$. *(b) The unstable* (W_u) *and stable* (W_s) *manifolds emanating from the periodic orbit P.*

The unstable manifold of P is defined as the set of all initial conditions $\mathbf{q}_0$ whose resulting orbits tend asymptotically to the periodic orbit in the backward sense of time. Namely:

$$W_u^P = \left\{ \mathbf{q}_0 : \lim_{t \to -\infty} d(\mathbf{q}(t; \mathbf{q}_0), P) = 0 \right\} . \quad (3)$$

The definition (3) implies that actually all orbits with initial conditions on W_u^P recede on average from the periodic orbit in the forward sense of time.

Furthermore, a straightforward consequence of the definition is that the set W_u^P is invariant under the phase flow, i.e. all initial conditions on W_u^P lead to orbits lying entirely on W_u^P.

Similarly, we define the stable manifold of P as the set of all initial conditions $\mathbf{q}_0$ whose resulting orbits tend asymptotically to the periodic orbit in the forward sense of time, i.e.

$$W_s^P = \left\{ \mathbf{q}_0 : \lim_{t \to \infty} d(\mathbf{q}(t; \mathbf{q}_0), P) = 0 \right\} . \quad (4)$$

The set W_s^P is also invariant under the phase flow of the Hamiltonian (2).

In numerical computations, the periodic orbit P can be found by a ‘root-finding’ algorithm (e.g. Newton’s one). We can also compute the eigenvalues and eigenvectors of the monodromy matrix of P, by solving numerically the variational equations of motion around P. Since P is unstable, the two eigenvalues (Λ_1, Λ_2) of the monodromy matrix are real and reciprocal. The unstable (stable) eigen-direction corresponds to the eigenvalue which is absolutely larger (smaller) that unity. In order to compute, say, the unstable manifold of P we take a small segment ΔS on the surface of section along the unstable eigen-direction, starting from the periodic orbit P, and compute the successive images of this segment under the surface of section mapping. In Fig. 3b, the unstable manifold is shown as a thin curve starting from the left side point P (which is the same as the right side point, modulo 2π), which has the form of a straight line close to P, but exhibits a number of oscillations as it approaches the right side point P. It should be noted that the possibility to obtain a picture of the manifold using an initial line segment relies on the so-called Grobman (1959) and Hartman (1960) theorem, which states that in a neighborhood of P the nonlinear flow around P is homeomorphic to the flow corresponding to the linearized equations of motion.

In a similar way we plot the stable manifold W_s^P emanating from P, taking an initial segment along the stable eigen-direction, and integrating in the backward sense of time. In Fig. 3b, the stable manifold is also shown by a thin curve, symmetric to the curve W_u^P with respect to the axis $\psi = 0$. This symmetry is a feature of the particular model under study.

Using the above example, we will now present the concept of the hyperbolic normal form, as well as how this can be used in computations related to unstable periodic orbits and their invariant manifolds.

The idea of a hyperbolic normal form is simple: close to any unstable periodic orbit, we wish to pass from old to new canonical variables $(\psi, \phi, p, I) \to (\xi, \phi', \eta, I')$, via a transformation of the form:

$$\begin{aligned} \psi &= \Phi_\psi(\xi, \phi', \eta, I') \\ \phi &= \Phi_\phi(\xi, \phi', \eta, I') \\ p &= \Phi_p(\xi, \phi', \eta, I') \\ I &= \Phi_I(\xi, \phi', \eta, I') \end{aligned} \quad (5)$$

so that the Hamiltonian in the new variables takes the form:

$$Z_h = \omega I' + \nu\xi\eta + Z(I,\xi\eta) \tag{6}$$

where ν is a real constant. In a Hamiltonian like (6), the point $\xi = \eta = 0$ corresponds to a periodic orbit, since we find $\dot{\xi} = \dot{\eta} = 0$ $\dot{I}' = 0$ from Hamilton's equations, while $\phi' = \phi_0' + (\omega + \partial Z(I',0)\partial I')t$. This implies a periodic orbit, with frequency $\omega' = (\omega + \partial Z(I',0)\partial I')$. Note that in a system like (2), where the action I is dummy, I' appears in the hyperbolic normal form only through the term $\omega I'$. Thus, the periodic solution $\xi = \eta = 0$ has a frequency always equal to ω.

By linearizing Hamilton's equations of motion near this solution, we find that it is always unstable. In fact, we can easily show that the linearized equations of motion for small variations $\delta\xi, \delta\eta$ around $\xi = 0, \eta = 0$ are:

$$\dot{\delta\xi} = (\nu + \nu_1(I))\delta\xi, \qquad \dot{\delta\eta} = -(\nu + \nu_1(I))\delta\eta$$

where $\nu_1(I) = \partial Z(I,\xi\eta = 0)/\partial(\xi\eta)$. The solutions are $\delta\xi(t) = \delta\xi_0 e^{(\nu+\nu_1)t}$, $\delta\eta(t) = \delta\eta_0 e^{-(\nu+\nu_1)t}$. After one period $T = 2\pi/\omega$ we have $\delta\xi(T) = \Lambda_1\delta\xi_0$, $\delta\eta(T) = \Lambda_2\delta\xi_0$, where $\Lambda_{1,2} = e^{\pm 2\pi(\nu+\nu_1)/\omega}$. Thus, the two eigendirections of the linearized flow correspond to setting $\delta\xi_0 = 0$, or $\delta\eta_0 = 0$, i.e. they coincide with the axes $\xi = 0$, or $\eta = 0$. These axes are invariant under the flow of (6) and, therefore, they constitute the unstable and stable manifold of the associated periodic orbit P in the new variables ξ, η.

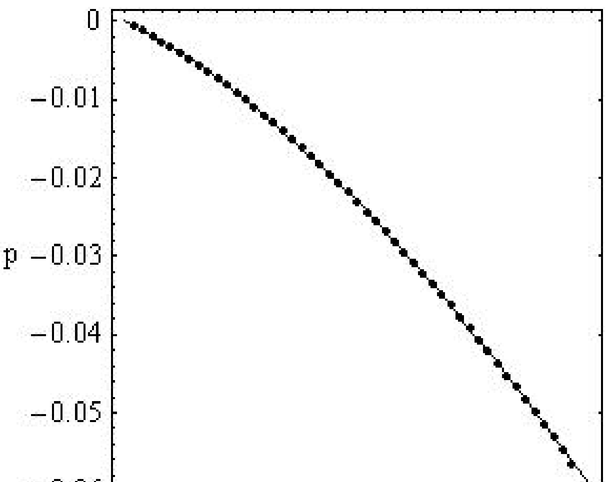

Fig. 4. *The characteristic curve (value of the fixed point variable p_P on the surface of section) for the main unstable periodic orbit as a function of ϵ. The dots correspond to a purely numerical calculation using Newton's method. The solid curve shows the theoretical calculation using a hyperbolic normal form (similar to formula (22) in text, but for a normalization up to the fifteenth order).*

After we explicitly compute the canonical transformations (5), Eqs. (5) can be used to compute analytically the periodic orbit and its asymptotic invariant manifolds in the original canonical variables as well. We will present the details of this computation in Section 3 below. However, we discuss now its outcome, shown in Figs. 4 and 5.

Fig. 4 shows the so-called *characteristic curve* of the unstable periodic orbit P. The characteristic curve yields the value of the initial conditions (on a surface of section) as a function of ϵ for which the resulting orbit is the periodic one. In our case, we always have $\psi_P = 0$ while p_P varies with ϵ. In order to compute $p_P(\epsilon)$ analytically, returning to Eqs. (5) we set $\xi = \eta = 0$. Furthermore, since I' is an integral of the Hamiltonian flow of (6), we replace its value by a constant $I' = c$. The value of c is fixed by the value of the energy at which the computation is done. Finally, knowing the frequency ω' by which ϕ' evolves, we can set $\phi' = \omega' t + \phi_0'$. Substituting these expressions in the transformation Eqs. (5), we are led to:

$$\begin{aligned} \psi_P(t) &= \Phi_\psi(0, \omega' t + \phi_0', 0, c) \\ \phi_P(t) &= \Phi_\phi(0, \omega' t + \phi_0', 0, c) \\ p_P(t) &= \Phi_p(0, \omega' t + \phi_0', 0, c) \\ I_P(t) &= \Phi_I(0, \omega' t + \phi_0', 0, c) \ . \end{aligned} \tag{7}$$

The set of Eqs. (7) yields now an analytic representation of the periodic orbit P in the whole time interval $0 \le t \le 2\pi/\omega'$. In fact, Eqs. (7) provide a formula for the periodic orbit in terms of Fourier series, which allows us to define not only its initial conditions on a surface of section, but also the time evolution for the whole set of canonical variables along P in the time interval $0 \le t \le T$.

As a comparison, the dotted curve in Fig. 4 shows $p_P(\epsilon)$ as computed by a purely numerical process, i.e., implementing Newton's root-finding method, while the solid curve yields $p_P(\epsilon)$ as computed by a hyperbolic normal form at the normalization order $r = 15$ (see below). The agreement is excellent, and we always recover 8-9 digits of the numerical calculation of the periodic orbit even for values of ϵ much larger than unity. In fact, since the origin is always included in the domain of convergence of the normal form, we can increase this accuracy by computing normal form approximations of higher and higher order.

Now, to compute the invariant manifolds of P by the normal form, we first fix a surface of section by setting, say, $\phi' = 0$. Let us assume without loss of generality that the unstable manifold corresponds to setting $\eta = 0$. Via the transformation equations, we then express all canonical variables as a function of ξ along the asymptotic curve of the unstable manifold on the surface of section, namely:

$$\psi_{P,u}(\xi) = \Phi_\psi(\xi, 0, 0, c), \quad p_{P,u}(\xi) = \Phi_p(\xi, 0, 0, c) \ . \tag{8}$$

Due to Eq. (8), ξ can be considered as a *length parameter* along the asymptotic curve of the unstable manifold W_u. Numerically, this allows to compute the asymptotic curve W_u on the surface of section by giving different values to ξ. Such a computation is shown by a thick curve in Fig. 5a. We observe that the theoretical curve W_u agrees well with the numerical one up to a certain distance corresponding to $\xi \sim 1$ whereby the theoretical curve starts deviating from the true asymptotic curve W_u. This is because, as discussed already, the hyperbolic normal form has a *finite domain of convergence* around P. Thus, by using a finite truncation of the series (5) (representing the normalizing canonical transformations), deviations occur at points beyond the domain of convergence of the hyperbolic normal form.

Similar arguments (and results, as shown in Fig. 5a) are found for the stable manifold of P. In that case, we substitute $\xi = 0$ in the transformation equations and employ η as a parameter, namely:

$$\psi_{P,s}(\eta) = \Phi_\psi(0,0,\eta,c), \quad p_{P,s}(\eta) = \Phi_p(0,0,\eta,c) \ . \tag{9}$$

In Fig. 5a we see that the domains of convergence of the hyperbolic normal form is small enough so that the two theoretical curves W_u and W_s have no intersection. This implies that we cannot use this computation to specify analytically the position of a homoclinic point, like H in Fig. 5. This corresponds to the case described in the schematic Figs. 2c and 2d.

However, Ozorio de Almeida and Viera (1997) have considered an extension of the original theory of Moser, which allows for a considerable extension of the domain of convergence of the hyperbolic normal form so as to include one or more homoclinic points. In this extension

i) we develop first the usual construction in order to compute analytically a finite segment of, say, W_u within the domain of convergence of the hyperbolic normal form. Then,

ii) we compute by analytic continuation one or more images of the initial segment, using to this end the original Hamiltonian as a Lie generating function of a symplectic transformation corresponding to the Poincaré mapping under the Hamiltonian flow of (2) itself. In the Appendix, we give the explicit formulae defining canonical transformations by Lie series. The final result can be stated as follows: If $\mathbf{q}$ is a point computed on the invariant manifold, we compute its image via:

$$\mathbf{q}' = \exp(t_n L_H)\exp(t_{n-1}L_H)\ldots\exp(t_1 L_H)\mathbf{q} \tag{10}$$

where $t_n + t_{n-1} + \ldots + t_1 = T$, while the times t_i are chosen so as to always lead to a mapping within the analyticity domain of the corresponding Lie series in a complex time domain. The Lie exponential operator in Eq. (10) is defined in the Appendix.

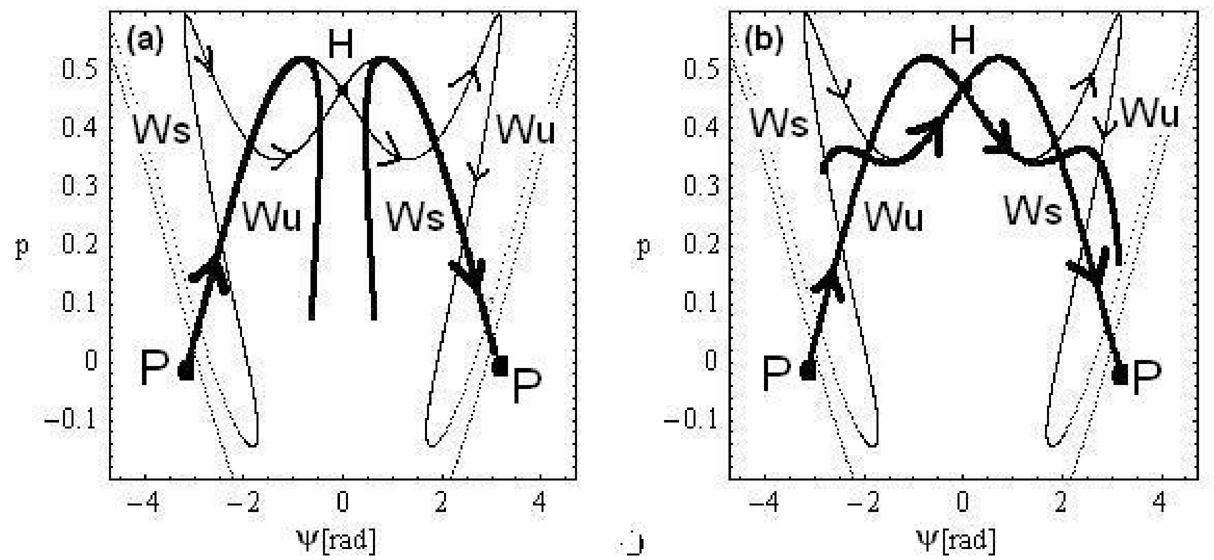

Fig. 5. *The thin dotted lines show the unstable (W_u) and stable ((W_s) manifolds emanating from the main unstable periodic orbit (P) in the model (2), for $\epsilon = 1$, after a purely numerical computation (mapping for 8 iterations of 1000 points along an initial segment of length $ds = 10^{-3}$ taken along the unstable and stable eigen-directions respectively. In (a), the thick lines show a theoretical computation of the invariant manifolds using a hyperbolic normal form at the normalization order $r = 15$ (see text). Both theoretical curves W_u and W_s deviate from the true manifolds before reaching the first homoclinic point (H). (b) Same as in (a) but now the theoretical manifolds are computed using the analytic continuation technique suggested in Ozorio de Almeida and Viera (1997). The theoretical curves cross each other at the first homoclinic point, thus, this point can be computed by series expansions.*

Fig. 5b shows the result obtained by applying Eq. (10) to the data on the invariant manifolds of Fig. 5a. In this computation we split the period $T = 2\pi$ in four equal time intervals of duration $t_1 = t_2 = t_3 = t_4 = \pi/2$. The thick lines show the theoretical computation of the images (for the unstable manifold), or pre-images (for the stable manifold, for which we put a minus sign in front of all times t_1 to t_4) of the thick lines shown in panel (a), after (or before) one period. We now see that the resulting series represent the true invariant manifolds over a considerably larger extent thus allowing to compute theoretically the position of the homoclinic point H.

3. DETAILS OF THE COMPUTATION

We now present in detail the steps leading to the previous results, i.e. a practical example of calculation of a hyperbolic normal form.

i) *Hamiltonian expansion.* Starting from the Hamiltonian (2) in the neighborhood of P (see phase portraits in Fig. 3) we first expand the Hamiltonian around the value $\psi_0 = \pi$ (or, equivalently, $-\pi$), which corresponds to the position of the unstable equilibrium when $\epsilon = 0$. Setting $\psi = \pi + u$, the first few terms (up to fourth order) are:

$$\begin{aligned} H &= \frac{p^2}{2} + I - 0.08\left(1 + 0.5\epsilon(1+p)(e^{i\phi} + e^{-i\phi})\right) \times \\ &\times \left(-1 + \frac{u^2}{2} - \frac{u^4}{24} - ...\right) . \end{aligned} \tag{11}$$

The hyperbolic character of motion in the neighborhood of the unstable equilibrium is manifested by the combination of terms:

$$H = I + \frac{p^2}{2} - 0.08\frac{u^2}{2} + ... \tag{12}$$

The constant ν appearing in Eq. (6) is related to the constant 0.08 appearing in Eq. (12) for $\nu^2 = 0.08$. In fact, if we write the hyperbolic part of the Hamiltonian as $H_h = p^2/2 - \nu^2 u^2/2$, it is possible to bring H_h in hyperbolic normal form by introducing a linear canonical transformation:

$$p = \frac{\sqrt{\nu}(\xi + \eta)}{\sqrt{2}}, \quad u = \frac{(\xi - \eta)}{\sqrt{2\nu}} \tag{13}$$

where ξ and η are the new canonical position and momentum respectively. Then H_h acquires the desired form, i.e. $H_h = \nu\xi\eta$.

Substituting the transformation (13) into the Hamiltonian (11) we find

$$\begin{aligned} H &= I + 0.282843\xi\eta - 0.041667\xi\eta^3 + 0.0625\xi^2\eta^2 - \\ &- 0.010417\xi^3\eta + 0.010417\xi^4 + \\ &+ \epsilon\Bigg[0.08 + 0.030085\eta - 0.070711\eta^2 - \\ &- 0.026591\eta^3 + 0.010417\eta^4 + 0.030085\xi + \\ &+ 0.14142\xi\eta + 0.0265915\xi\eta^2 - 0.041667\xi\eta^3 - \\ &- 0.070711\xi^2 + 0.026591\xi^2\eta + 0.0625\xi^2\eta^2 - \\ &- 0.026591\xi^3 - 0.041667\xi^3\eta + \\ &+ 0.010417\xi^4 + ...\Bigg]\left(\frac{e^{i\phi} + e^{-i\phi}}{2}\right) . \end{aligned}$$

In computer-algebraic calculations, it is now convenient to introduce an artificial parameter λ, with numerical value equal to $\lambda = 1$, called the 'book-keeping parameter' (see Efthymiopoulos 2008). We put a factor λ^r in front of each term in the above Hamiltonian expansion which indicates that the term is to be considered at the r-th normalization step. Furthermore, we carry λ in all subsequent algebraic operations. In this way, we can keep track of the estimated order of smallness of each term which either exists in the original Hamiltonian or is generated in the course of the normalization process.

In the present case, it is crucial to recognize that the quantities ξ, η themselves can be considered as small quantities describing the neighborhood of a hyperbolic point. For reasons explained below, we want to retain a book-keeping factor λ^0 for the lowest order term $\xi\eta$. We thus impose the rule that monomial terms containing a product $\xi^{s_1}\eta^{s_2}$ acquire a book-keeping factor $\lambda^{s_1+s_2-2}$ in front. Finally, we add a book-keeping factor λ to all the terms that are multiplied by ϵ.

After the introduction of the book-keeping parameter, up to $O(\lambda^2)$ the Hamiltonian reads:

$$\begin{aligned} H^{(0)} &= I + 0.282843\xi\eta + \lambda\epsilon\Bigg[0.04 + \\ &+ 0.0150424(\xi + \eta) - 0.0353553(\xi^2 + \eta^2) + \\ &+ 0.0707107\xi\eta\Bigg](e^{i\phi} + e^{-i\phi}) + \\ &+ \lambda^2\Bigg[0.0104167(\xi^4 + \eta^4) - \\ &- 0.0416667(\xi\eta^3 + \xi^3\eta) + 0.0625\xi^2\eta^2 + \\ &+ 0.0132957\epsilon(\xi^2\eta + \xi\eta^2 - \xi^3 - \eta^3)\Bigg] \cdot \\ &\cdot \quad (e^{i\phi} + e^{-i\phi}) + \ldots \end{aligned}$$

ii) *Hamiltonian normalization.* The aim of the hamiltonian normalization is to define a sequence of near-identity canonical transformations:

$$\begin{aligned} (\xi, \eta, \phi, I) &\equiv (\xi^{(0)}, \eta^{(0)}, \phi^{(0)}, I^{(0)}) \rightarrow \\ &\rightarrow (\xi^{(1)}, \eta^{(1)}, \phi^{(1)}, I^{(1)}) \rightarrow \\ &\rightarrow (\xi^{(2)}, \eta^{(2)}, \phi^{(2)}, I^{(2)}) \rightarrow \ldots \end{aligned}$$

such that the original Hamiltonian $H \equiv H^{(0)}$ is transformed to $H^{(1)}$, $H^{(2)}$,... respectively, with the property that after r steps, the Hamiltonian $H^{(r)}$ is in normal form, according to the definition (6), up to terms of order $O(\lambda^r)$.

The normalization can by accomplished by means of Lie series (see the Appendix) via the following recursive algorithm. After r steps, the Hamiltonian has the form:

$$\begin{aligned} H^{(r)} &= Z_0 + \lambda Z_1 + ... + \lambda^r Z_r + \lambda^{r+1} H^{(r)}_{r+1} + \\ &+ \lambda^{r+2} H^{(r)}_{r+2} + \dots \end{aligned} \tag{14}$$

where $Z_0 = \omega I + \nu\xi\eta$. The Hamiltonian term $H^{(r)}_{r+1}$ contains some terms that are not in normal form according to the definition (6). Denoting the ensemble of these terms by $h^{(r)}_{r+1}$, we compute the Lie generating function χ_{r+1} as the solution of the homological equation:

$$\{Z_0, \chi_{r+1}\} + \lambda^{r+1} h^{(r)}_{r+1} = 0 \tag{15}$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket operator. We then compute the new transformed Hamiltonian via:

$$H^{(r+1)} = \exp(L_{\chi_{r+1}}) H^{(r)} \ . \tag{16}$$

This is in normal form up to terms of order $r+1$, namely:

$$\begin{aligned} H^{(r+1)} &= Z_0 + \lambda Z_1 + ... + \lambda^r Z_r + \lambda^{r+1} Z_{r+1} + \\ &+ \lambda^{r+2} H^{(r+1)}_{r+2} + \dots \end{aligned} \tag{17}$$

where $Z_{r+1} = H^{(r)}_{r+1} - h^{(r)}_{r+1}$.

The solution of the homological equation is readily found by noting that the action of the operator $\{Z_0,\cdot\} = \{\omega I + \nu\xi\eta,\cdot\}$ on monomials of the form $\xi^{s_1}\eta^{s_2} a(I) e^{ik_2\phi}$ yields:

$$\begin{aligned} &\left\{\omega I + \nu\xi\eta, \xi^{s_1}\eta^{s_2} a(I) e^{ik_2\phi}\right\} = \\ &-[(s_1-s_2)\nu + i\omega k_2]\xi^{s_1}\eta^{s_2} a(I) e^{ik_2\phi} \ . \end{aligned}$$

Thus, if we write $h^{(r)}_{r+1}$ as:

$$h^{(r)}_{r+1} = \sum_{(s_1,s_2,k_2)\notin\mathcal{M}} b_{s_1,s_2,k_2}(I)\xi^{s_1}\eta^{s_2} e^{ik_2\phi}$$

where $\mathcal{M}$ denotes the so-called *resonant module* defined by:

$$\mathcal{M} = \{(s_1,s_2,k_2) : s_1 = s_2 \text{ and } k_2 = 0\}\,, \tag{18}$$

then the solution of the homological equation (15) is:

$$\chi_1 = \sum_{(s_1,s_2,k_2)\notin\mathcal{M}} \frac{b_{s_1,s_2,k_2}(I)}{(s_1-s_2)\nu + i\omega k_2} \xi^{s_1}\eta^{s_2} e^{ik_2\phi} \ . \tag{19}$$

The main remark regarding Eq. (19) is that the divisors are complex numbers with a modulus bounded from below by a positive constant, i.e. we have:

$$\begin{aligned} &|\nu(s_1-s_2) + ik_2\omega| = \\ = \ &\sqrt{(s_1-s_2)^2\nu^2 + k_2^2\omega^2} \geq \min(|\nu|,|\omega|) \\ &\text{for all } (s_1,s_2,k_2) \notin \mathcal{M} \ . \end{aligned} \tag{20}$$

This last bound constitutes the most relevant fact about the hyperbolic normal form construction because it implies that this construction is convergent with a finite analyticity domain at the limit $r\to\infty$. A formal proof of this fact is given in Giorgilli (2001).

As an example, returning to our computations regarding the specific model of Figs. 3 to 5, we will present the detailed computation of the hyperbolic normal form of order $O(\lambda)$. Note a simplification in the notation below, i.e. that we omit superscripts of the form $^{(r)}$ for all the canonical variables, keeping such superscripts only in the various quantities depending on these variables.

According to the general algorithm, at first order we want to eliminate i) terms depending on the angle ϕ, or, ii) terms independent of ϕ but depending on a product $\xi^{s_1}\eta^{s_2}$ with $s_1 \neq s_2$. These are:

$$\begin{aligned} h_1^{(0)} = \epsilon \Bigg[&0.04 + 0.0150424(\xi+\eta) - \\ &- 0.0353553(\xi^2+\eta^2) + 0.0707107\xi\eta \Bigg](e^{i\phi} + e^{-i\phi})\Bigg] \ . \end{aligned}$$

The homological equation defining the generating function χ_1 is given by:

$$\{I + 0.282843\xi\eta, \chi_1\} + \lambda h^{(0)} = 0 \ . \tag{21}$$

Following Eq. (19), the solution of Eq. (21) is:

$$\begin{aligned} \chi_1 &= \lambda\epsilon i\Bigg[\Bigg(-0.04 + (0.00393948 - 0.0139282i)\xi - \\ &- (0.00393948 + 0.0139282i)\eta - (0.0151515 - \\ &- 0.0267843i)\xi^2 + (0.0151515 + 0.0267843i)\eta^2 - \\ &- 0.070711\xi\eta\Bigg)e^{i\phi} + \\ &+ \Bigg(0.04 + (0.00393948 + 0.0139282i)\xi - \\ &- (0.00393948 - 0.0139282i)\eta - (0.0151515 + \\ &+ 0.0267843i)\xi^2 + (0.0151515 - 0.0267843i)\eta^2 + \\ &+ 0.070711\xi\eta\Bigg)e^{-i\phi}\Bigg] \ . \end{aligned}$$

The normalized Hamiltonian, after computing $H^{(1)} = \exp(L_{\chi_1})H^{(0)}$ is in normal form up to terms of $O(\lambda)$. In fact, we find that there are no new normal form terms at this order, but such terms appear at order λ^2. Computing, in the same way as above, the generating function χ_2, we find $H^{(2)} =$

$\exp(L_{\chi_2})H^{(1)}$, in normal form up to order two. This is

$$\begin{aligned} H^{(2)} &= I + 0.282843\eta\xi + \lambda^2(0.0625\xi^2\eta^2 - \\ &- \epsilon^2 0.0042855\xi\eta) + O(\lambda^3) + \ldots \end{aligned}$$

Higher order normalization requires use of a computer-algebraic program since the involved operations soon become quite cumbersome.

For completeness we give below the analytic expression for the periodic orbit P up to order $O(\lambda^2)$ found as explained above, i.e. by exploiting the normalizing transformations of the hyperbolic normal form. The old canonical variables (ξ,η) are computed in terms of the new canonical variables $(\xi^{(2)},\eta^{(2)})$ following:

$$\xi = \exp(L_{\chi_2})\exp(L_{\chi_1})\xi^{(2)}$$

$$\eta = \exp(L_{\chi_2})\exp(L_{\chi_1})\eta^{(2)} ~.$$

This yields functions (up to order $O(\lambda^2)$) $\xi = \Phi_\xi(\xi^{(2)},\phi^{(2)},\eta^{(2)})$, and $\eta = \Phi_\eta(\xi^{(2)},\phi^{(2)},\eta^{(2)})$. By virtue of the fact that I is a dummy action, we have $\phi^{(2)} = \phi = \omega t = t$ while we set $\xi^{(2)} = \eta^{(2)} = 0$ for the periodic orbit. With these substitutions we find:

$$\xi_P(t) = \Phi_\xi(0,t,0), \quad \eta_P(t) = \Phi_\eta(0,t,0) ~.$$

Finally, we substitute the expressions for $\xi_P(t)$ and $\eta_P(t)$ in the linear canonical transformation (13), in order to find analytic expressions for the periodic orbit in the original variables $p, \psi = \pi + u$. Switching back to trigonometric functions and setting $\lambda = 1$, we finally find:

$$\begin{aligned} \psi_P(t) &= \pi + 0.0740741\epsilon\sin t - \\ &- 0.000726216\epsilon^2\sin(2t) + \ldots \qquad (22) \\ p_P(t) &= -0.00592593\epsilon\cos t - \\ &- 0.00145243\epsilon^2\cos(2t) + \ldots ~. \end{aligned}$$

The position of the periodic orbit on the surface of section can now be found by setting $t = 0$ in Eqs. (22). In the actual computation of Figs. 4 and 5, we compute all expansions up to $O(\lambda^{15})$, after expanding also $\cos\psi$ in the original Hamiltonian up to the same order.

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APPENDIX: CANONICAL TRANSFORMATIONS BY LIE SERIES

The use of Lie transformations in canonical perturbation theory was introduced by Hori (1966) and Deprit (1969). Let us consider an arbitrary function $\chi(\psi,\phi,p,I)$ and compute the Hamiltonian flow of χ given by:

$$\dot\psi=\frac{\partial\chi}{\partial p},\quad \dot\phi=\frac{\partial\chi}{\partial I},\quad \dot p=-\frac{\partial\chi}{\partial\psi},\quad \dot I=-\frac{\partial\chi}{\partial\phi}\ . \quad (23)$$

Let $\psi(t),\phi(t),p(t),I(t)$ be a solution of Eqs. (23) for some choice of initial conditions $\psi(0)=\psi_0$, $\phi(0)=\phi_0$, $p(0)=p_0$, and $I(0)=I_0$. For any time t, the mapping of the variables in time, namely:

$$(\psi_0,\phi_0,p_0,I_0)\to(\psi_t,\phi_t,p_t,I_t)$$

can be proven to be a canonical transformation (see, for example, Arnold (1978)). In that sense, any arbitrary function $\chi(\psi,\phi,p,I)$ can be thought of as a function which can generate an infinity of different canonical transformations via its Hamilton equations of motion solved for infinitely many different values of time t. The function χ is called a *Lie generating function*.

Consider now the Poisson bracket operator $L_\chi\equiv\{\cdot,\chi\}$ whose action on functions $f(\psi,\phi,p,I)$ is defined by:

$$L_\chi f=\{f,\chi\}=\frac{\partial f}{\partial\psi}\frac{\partial\chi}{\partial p}+\frac{\partial f}{\partial\phi}\frac{\partial\chi}{\partial I}-\frac{\partial f}{\partial p}\frac{\partial\chi}{\partial\psi}-\frac{\partial f}{\partial I}\frac{\partial\chi}{\partial\phi}\ . \quad (24)$$

The time derivative of any function $f(\psi,\phi,p,I)$ along a Hamiltonian flow defined by the function χ is given by:

$$\begin{aligned}\frac{df}{dt}&=\frac{\partial f}{\partial\psi}\dot\psi+\frac{\partial f}{\partial\phi}\dot\phi+\frac{\partial f}{\partial p}\dot p+\frac{\partial f}{\partial I}\dot I=\\&=\frac{\partial f}{\partial\psi}\frac{\partial\chi}{\partial p}+\frac{\partial f}{\partial\phi}\frac{\partial\chi}{\partial I}-\frac{\partial f}{\partial p}\frac{\partial\chi}{\partial\psi}-\frac{\partial f}{\partial I}\frac{\partial\chi}{\partial\phi}\ ,\end{aligned}$$

that is:

$$\frac{df}{dt}=\{f,\chi\}=L_\chi f\ . \quad (25)$$

Extending this to higher order derivatives, we have

$$\frac{d^nf}{dt^n}=\{\ldots\{\{f,\chi\},\chi\}\ldots\chi\}=L_\chi^n f\ . \quad (26)$$

Writing the solution of, say ψ_t, for a given set of initial conditions as a Taylor series:

$$\psi_t=\psi_0+\frac{d\psi_0}{dt}t+\frac{d^2\psi_0}{dt^2}t^2+\ldots=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{d^n\psi_0}{dt^n}t^n\ , \quad (27)$$

and taking into account that the Taylor expansion of the exponential around the origin is given by

$$\exp(x)=1+x+\frac{x^2}{2}+\frac{x^3}{3!}+\ldots=\sum_{n=0}^{\infty}\frac{x^n}{n!}$$

we can see that the Taylor expansion (27) is formally given by the following exponential operator:

$$\exp\frac{d}{dt}=1+\frac{d}{dt}+\frac{1}{2}\frac{d^2}{dt^2}+\ldots$$

Taking into account Eqs. (25) and (26), we are lead to the formal definition of the *Lie series*:

$$\psi_t=\psi_0+(L_\chi\psi_0)t+\frac{1}{2}(L_\chi^2\psi_0)t^2+\ldots \quad (28)$$

Setting, finally, the time as $t=1$, we arrive at the formal definition of a canonical transformation using Lie series by:

$$\begin{aligned}&\psi_1=\exp(L_\chi)\psi_0,\quad \phi_1=\exp(L_\chi)\phi_0,\\&p_1=\exp(L_\chi)p_0,\quad I_1=\exp(L_\chi)I_0\ .\end{aligned} \quad (29)$$

A basic property of Lie transformations is that the change in the form of an arbitrary function f of a set of canonical variables under a Lie transformation can be found by acting directly with the Lie operator $\exp(L_\chi)$ on f, i.e.:

$$\begin{aligned}&f(\exp(L_\chi)\psi,\exp(L_\chi)\phi,\exp(L_\chi)p,\exp(L_\chi)I)=\\&\quad=\exp(L_\chi)f(\psi,\phi,p,I)\ .\end{aligned} \quad (30)$$

Thus, computations of canonical perturbation theory based on Lie transformations involve only the evaluation of derivatives, which is a straightforward algorithmic procedure. This fact renders the method of Lie transformations quite convenient for the implementation of computer-algebraic computations of normal forms.

ХИПЕРБОЛИЧКЕ НОРМАЛНЕ ФОРМЕ И ИНВАРИЈАНТНЕ МНОГОСТРУКОСТИ. ПРИМЕНЕ У АСТРОНОМИЈИ

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Прегледни рад по позиву

Последњих година проучавање динамике нестабилних периодичних орбита на инваријантним многострукостима, код нелинеарних Хамилтонових динамичких система, довело је до бројних примена у небеској механици и динамичкој астрономији. Две тренутно најзначајније примене су i) у свемирској механици на многострукостима, тј. коришћење многострукости приликом дизајнирања свемирских мисија, и, у потпуно другачијем контексту, ii) за проучавање спиралне структуре галаксија. У данашње време већина приступа за израчунавање орбита повезаних са динамиком на многострукостима (тј. периодичним или асимптотским орбитама) ослања се, или на тзв. Poinkare-Lindstedt методу, или на чисто нумеричка израчунавања. У овом раду дајемо кратак приказ једне аналитичке методе за одређивање инваријантне многострукости, првобитно предложене од стране Мозера (Moser 1958), а касније развијене у канонском облику од стране Ђорђилија (Giorgilli 2001). Користимо једноставан пример за демонстрацију како се може извршити одређивање хиперболичке нормалне форме, позивајући се на аналитичко проширење методе од стране Озориа де Алмеиде и коаутора, помоћу којег можемо значајно продужити иницијални домен конвергенције Мозерове нормалне форме.