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ZERNIKE BASIS TO CARTESIAN TRANSFORMATIONS

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SUMMARY: The radial polynomials of the 2D (circular) and 3D (spherical) Zernike functions are tabulated as powers of the radial distance. The reciprocal tabulation of powers of the radial distance in series of radial polynomials is also given, based on projections that take advantage of the orthogonality of the polynomials over the unit interval. They play a role in the expansion of products of the polynomials into sums, which is demonstrated by some examples.
Multiplication of the polynomials by the angular bases (azimuth, polar angle) defines the Zernike functions, for which we derive transformations to and from the Cartesian coordinate system centered at the middle of the circle or sphere.

Key words. Instrumentation: adaptive optics – Methods: analytical

1. SCOPE

The 2D Zernike functions are an orthogonal basis over the circle, which is a pupil section of an optical system in the majority of applications. Orthogonality is helpful if Strehl ratios are derived from expansion coefficients, for example, supposed phases of wave fronts perturbed by turbulence are expanded in such a basis.

Most of the uses of the 3D Zernike functions have been hashes of shape characteristics for pattern recognition (Mak et al. 2008). The incentive of this work was modeling of ray paths through a turbulent atmosphere, if the dielectric function is expanded in a Zernike basis inside a sphere (Mathar 2008). Since calculation of gradients is more tedious in curvilinear local spherical coordinates than in Cartesian coordinates, a transformation of the basis is useful.

The Zernike radial polynomials $R_n^{(l)}(r)$ are embedded in an orthogonal system in the unit sphere (Bhatia and Wolf 1952, 1954) of integer dimension $D \geq 1$,

$$\int_0^1 r^{D-1} R_n^{(l)} R_{n'}^{(l)} dr = \delta_{n,n'} \tag{1}$$

for even $n-l$ and $0 \leq l \leq n$. [The normalization in two dimensions (9) will usually be chosen differently.] A set of spherical Harmonics for the $D-1$ angular degrees of freedom complements the basis (Blumenson 1960, Ehrentraud and Muschik 2004). The ansatz $R_n^{(l)} = r^l X_n^{(l)}(r^2)$ reveals that $X_n^{(l)}(z)$ is proportional to the Courant-Hilbert Jacobi Polynomial $G_{(n-l)/2}(l+D/2, l+D/2, z)$ (22.2.2 and 22.3.3 of Abramowitz and Stegun 1972) and normalization leads to

$$R_n^{(l)} = \sqrt{2n+D}\sum_{s=0}^{\frac{n-l}{2}}(-1)^s\binom{\frac{n-l}{2}}{s} \times\binom{\frac{D}{2}+n-s-1}{\frac{n-l}{2}}r^{n-2s}$$
$$= (-1)^{\frac{n-l}{2}}\sqrt{2n+D}r^l\sum_{s=0}^{\frac{n-l}{2}}(-1)^s\binom{\frac{n-l}{2}}{s} \times\binom{\frac{D}{2}+\frac{n+l}{2}+s-1}{\frac{n-l}{2}}r^{2s}$$
$$= (-1)^{\frac{n-l}{2}}\sqrt{2n+D}\binom{\frac{D}{2}+\frac{n+l}{2}-1}{\frac{n-l}{2}}r^l \times {}_2F_1\left(-\frac{n-l}{2},\frac{D}{2}+\frac{n+l}{2};\frac{D}{2}+l;r^2\right), \qquad (2)$$

where ${}_2F_1$ is the Gaussian Hypergeometric Function. Time-dependent turbulence with time along the fourth axis (mediated by some constant velocity) is a prospective case in $D = 4$ dimensions.

There is sufficient similarity between the 3D and 2D Zernike functions to present both cases conjointly. The more familiar 2D Zernike functions in Noll's nomenclature are covered in Section 2—although no new aspects beyond earlier work emerge (Dai 2006)—and similar methodology is independently applied to the 3D case in Section 3.

2. ZERNIKE CIRCLE FUNCTIONS

2.1. Zernike Radial Polynomials

We define Zernike radial polynomials in Noll's nomenclature (Noll 1976, Prata and Rusch 1989, Kintner 1976, Tyson 1982, Conforti 1983, Tango 1977, Mathar 2007) for $0 \le m \le n$, even $n-m$, as

$$R_n^m(r) = \sum_{s=0}^{\frac{n-m}{2}}(-1)^s\binom{n-s}{s}\binom{n-2s}{\frac{n-m}{2}-s}r^{n-2s} \qquad (3)$$
$$= (-1)^{\frac{n-m}{2}}r^m\sum_{s=0}^{\frac{n-m}{2}}\binom{\frac{n+m}{2}+s}{\frac{n-m}{2}-s}\binom{m+2s}{s}(-r^2)^s \qquad (4)$$
$$= (-1)^a\binom{-b}{-a}r^m\,{}_2F_1\left(a,1-b;1+m;r^2\right) \qquad (5)$$
$$= \binom{n}{-a}r^n\,{}_2F_1\left(a,b;-n;\frac{1}{r^2}\right), \qquad (6)$$

where a set of two negative parameters

$$a \equiv -(n-m)/2; \quad b \equiv -(n+m)/2 \qquad (7)$$

abbreviates the notation in the context of hypergeometric series. Following the original notation, we shall not put parentheses around the upper index m in R_n^m—although it is not a power.

Table 1. Examples of (3).

$R_0^0(r) = 1.$
$R_1^1(r) = r.$
$R_2^0(r) = -1+2r^2.$
$R_2^2(r) = r^2.$
$R_3^1(r) = -2r+3r^3.$
$R_3^3(r) = r^3.$
$R_4^0(r) = 1-6r^2+6r^4.$
$R_4^2(r) = -3r^2+4r^4.$
$R_4^4(r) = r^4.$
$R_5^1(r) = 3r-12r^3+10r^5.$
$R_5^3(r) = -4r^3+5r^5.$
$R_5^5(r) = r^5.$
$R_6^0(r) = -1+12r^2-30r^4+20r^6.$
$R_6^2(r) = 6r^2-20r^4+15r^6.$
$R_6^4(r) = -5r^4+6r^6.$
$R_6^6(r) = r^6.$
$R_7^1(r) = -4r+30r^3-60r^5+35r^7.$
$R_7^3(r) = 10r^3-30r^5+21r^7.$
$R_7^5(r) = -6r^5+7r^7.$
$R_7^7(r) = r^7.$
$R_8^0(r) = 1-20r^2+90r^4-140r^6+70r^8.$
$R_8^2(r) = -10r^2+60r^4-105r^6+56r^8.$
$R_8^4(r) = 15r^4-42r^6+28r^8.$
$R_8^6(r) = -7r^6+8r^8.$
$R_8^8(r) = r^8.$

The inversion of (4) decomposes powers r^j into sums of $R_n^m(r)$,

$$r^j \equiv \sum_{n=m \bmod 2}^{j} h_{j,n,m}R_n^m(r);\ j-m = 0,2,4,\dots \qquad (8)$$

Multiplying this equation by $R_{n'}^m(r)$, performing contraction on the right hand side with the normalization integral

$$\int_0^1 r\,R_n^m(r)R_{n'}^m(r)dr = \frac{1}{2(n+1)}\delta_{n,n'}, \qquad (9)$$

and using the explicit polynomial expression (4) on the left hand side yields (Mathar 2007, Sánchez-Ruiz and Dehesa 1997):

$$h_{j,n,m} = (n+1)(-1)^a\frac{\left(\frac{m-j}{2}\right)_{-a}}{\left(1+\frac{m+j}{2}\right)_{1-a}}. \qquad (10)$$

The parentheses with an integer index are Pochhammer's symbol defined as

$$(x)_t \equiv x(x+1)(x+2)\cdots(x+t-1);(x)_0 \equiv 1. \qquad (11)$$

From

$$R_n^m(1) = 1 \qquad (12)$$

and (8) at $r = 1$, we obtain the sum rule

$$\sum_{n=m \bmod 2}^{j} h_{j,n,m} = 1. \tag{13}$$

Basic properties of the Gamma Functions—implicitly contained in (10)— exhibit recurrences:

$$h_{j+2,n,m} = \frac{(j+2+m)(j+2-m)}{(j+2-n)(j+4+n)} h_{j,n,m}; \tag{14}$$

$$h_{j,n+2,m} = \frac{(n+3)(j-n)}{(j+4+n)(n+1)} h_{j,n,m}; \tag{15}$$

$$h_{j,n,m+2} = \frac{j+2+m}{j-m} h_{j,n,m}. \tag{16}$$

Table 2. Examples of (8).

$$\begin{aligned}
r^0 &= R_0^0(r).\\
r^1 &= R_1^1(r).\\
r^2 &= \tfrac{1}{2}R_0^0(r) + \tfrac{1}{2}R_2^0(r)\\
&= R_2^2(r).\\
r^3 &= \tfrac{2}{3}R_1^1(r) + \tfrac{1}{3}R_3^1(r)\\
&= R_3^3(r).\\
r^4 &= \tfrac{1}{3}R_0^0(r) + \tfrac{1}{2}R_2^0(r) + \tfrac{1}{6}R_4^0(r)\\
&= \tfrac{3}{4}R_2^2(r) + \tfrac{1}{4}R_4^2(r)\\
&= R_4^4(r).\\
r^5 &= \tfrac{1}{2}R_1^1(r) + \tfrac{2}{5}R_3^1(r) + \tfrac{1}{10}R_5^1(r)\\
&= \tfrac{4}{5}R_3^3(r) + \tfrac{1}{5}R_5^3(r)\\
&= R_5^5(r).\\
r^6 &= \tfrac{1}{4}R_0^0(r) + \tfrac{9}{20}R_2^0(r) + \tfrac{1}{4}R_4^0(r) + \tfrac{1}{20}R_6^0(r)\\
&= \tfrac{3}{5}R_2^2(r) + \tfrac{1}{3}R_4^2(r) + \tfrac{1}{15}R_6^2(r)\\
&= \tfrac{5}{6}R_4^4(r) + \tfrac{1}{6}R_6^4(r)\\
&= R_6^6(r).\\
r^7 &= \tfrac{2}{5}R_1^1(r) + \tfrac{2}{5}R_3^1(r) + \tfrac{6}{35}R_5^1(r) + \tfrac{1}{35}R_7^1(r)\\
&= \tfrac{2}{3}R_3^3(r) + \tfrac{2}{7}R_5^3(r) + \tfrac{1}{21}R_7^3(r)\\
&= \tfrac{6}{7}R_5^5(r) + \tfrac{1}{7}R_7^5(r)\\
&= R_7^7(r).\\
r^8 &= \tfrac{1}{5}R_0^0(r) + \tfrac{2}{5}R_2^0(r) + \tfrac{2}{7}R_4^0(r) + \tfrac{1}{10}R_6^0(r) + \tfrac{1}{70}R_8^0(r)\\
&= \tfrac{1}{2}R_2^2(r) + \tfrac{5}{14}R_4^2(r) + \tfrac{1}{8}R_6^2(r) + \tfrac{1}{56}R_8^2(r)\\
&= \tfrac{5}{7}R_4^4(r) + \tfrac{1}{4}R_6^4(r) + \tfrac{1}{28}R_8^4(r)\\
&= \tfrac{7}{8}R_6^6(r) + \tfrac{1}{8}R_8^6(r)\\
&= R_8^8(r).
\end{aligned}$$

2.2. Zernike Functions

The Zernike functions $Z_j(r,\varphi)$ are products of the radial polynomials by azimuth functions, $\cos(m\varphi)$ or $\sin(m\varphi)$, associated with Cartesian coordinates

$$x = r\cos\varphi; \quad y = r\sin\varphi. \tag{17}$$

Starting at $Z_1 = 1$, the index j is increased by increasing m, increasing n, when this is exhausted as $n = m$ has been reached, keeping j even for functions $\propto \cos(m\varphi)$, $m > 0$ and keeping j odd for functions $\propto \sin(m\varphi)$ (Noll 1976), we get:

$$Z_j = \begin{cases} \sqrt{2n+2}\, R_n^m(r)\cos(m\varphi), & m > 0, j \text{ even};\\ \sqrt{2n+2}\, R_n^m(r)\sin(m\varphi), & m > 0, j \text{ odd};\\ \sqrt{n+1}\, R_n^m(r), & m = 0. \end{cases} \tag{18}$$

$$\int_{r<1} Z_j(r,\varphi) Z_k(r,\varphi) d^2r = \pi\delta_{j,k};\ d^2r = r\,dr d\varphi. \tag{19}$$

Table 3. Examples of (18).

$$\begin{aligned}
Z_1 &= R_0^0(r).\\
Z_2 &= 2R_1^1(r)\cos(\varphi).\\
Z_3 &= 2R_1^1(r)\sin(\varphi).\\
Z_4 &= \sqrt{3}\,R_2^0(r).\\
Z_5 &= \sqrt{6}\,R_2^2(r)\sin(2\varphi).\\
Z_6 &= \sqrt{6}\,R_2^2(r)\cos(2\varphi).\\
Z_7 &= 2\sqrt{2}\,R_3^1(r)\sin(\varphi).\\
Z_8 &= 2\sqrt{2}\,R_3^1(r)\cos(\varphi).\\
Z_9 &= 2\sqrt{2}\,R_3^3(r)\sin(3\varphi).\\
Z_{10} &= 2\sqrt{2}\,R_3^3(r)\cos(3\varphi).\\
Z_{11} &= \sqrt{5}\,R_4^0(r).\\
Z_{12} &= \sqrt{10}\,R_4^2(r)\cos(2\varphi).\\
Z_{13} &= \sqrt{10}\,R_4^2(r)\sin(2\varphi).\\
Z_{14} &= \sqrt{10}\,R_4^4(r)\cos(4\varphi).\\
Z_{15} &= \sqrt{10}\,R_4^4(r)\sin(4\varphi).\\
Z_{16} &= 2\sqrt{3}\,R_5^1(r)\cos(\varphi).
\end{aligned}$$

2.3. Cartesian to Zernike

From (17), each multinomial x^py^q may be decomposed into terms proportional to $R_n^m(r)\cos(m\varphi)$ if q is even, or proportional to $R_n^m(r)\sin(m\varphi)$ if q is odd,

$$x^py^q = r^j\cos^p\varphi\sin^q\varphi, \quad j \equiv p+q, \tag{20}$$

$$\cos^p\varphi\sin^q\varphi = \frac{(-1)^{\lfloor q/2\rfloor}}{2^j} \tag{21}$$

$$\times \begin{cases} \Big[2\sum_{m=0}^{\lfloor (j-1)/2\rfloor}\sum_{l=\max(0,m-p)}^{\min(q,m)}\binom{p}{m-l}\binom{q}{l} \\ \times(-1)^l\cos[(j-2m)\varphi] + C(p,q)\Big], & q \text{ even};\\ 2\sum_{m=0}^{\lfloor (j-1)/2\rfloor}\sum_{l=\max(0,m-p)}^{\min(q,m)}\binom{p}{m-l}\binom{q}{l} \\ \times(-1)^l\sin[(j-2m)\varphi], & q \text{ odd}; \end{cases}$$

where

$$C(p,q) \equiv \begin{cases} 2^j\frac{\Gamma((p+1)/2)}{\Gamma(j/2+1)\Gamma((1-q)/2)} \\ = (-1)^{q/2}2^{j/2}\frac{(q-1)!!(p-1)!!}{(j/2)!}, & j \text{ even};\\ 0, & j \text{ odd} \end{cases}$$

is a "dangling" component representing the contribution at $j = 2m$ if q and j are both even (Dai 1995). The notation $\lfloor x \rfloor$ is the floor function, the largest integer $\le x$.

This extends Section 1.32 of the Gradstein-Ryshik (1981) tables, and distributes $\cos^p\varphi \sin^q\varphi$ into sums over $\cos(m\varphi)$ or $\sin(m\varphi)$ as in Table 4.

Table 4. Examples of (22).

$\cos^2\varphi$	$= [1 + \cos(2\varphi)]/2.$
$\cos\varphi\sin\varphi$	$= [\sin(2\varphi)]/2.$
$\sin^2\varphi$	$= [1 - \cos(2\varphi)]/2.$
$\cos^3\varphi$	$= [3\cos(\varphi) + \cos(3\varphi)]/4.$
$\cos^2\varphi\sin\varphi$	$= [\sin(\varphi) + \sin(3\varphi)]/4.$
$\cos\varphi\sin^2\varphi$	$= [\cos(\varphi) - \cos(3\varphi)]/4.$
$\sin^3\varphi$	$= [3\sin(\varphi) - \sin(3\varphi)]/4.$
$\cos^4\varphi$	$= [3 + 4\cos(2\varphi) + \cos(4\varphi)]/8.$
$\cos^3\varphi\sin\varphi$	$= [2\sin(2\varphi) + \sin(4\varphi)]/8.$
$\cos^2\varphi\sin^2\varphi$	$= [1 - \cos(4\varphi)]/8.$
$\cos\varphi\sin^3\varphi$	$= [2\sin(2\varphi) - \sin(4\varphi)]/8.$
$\sin^4\varphi$	$= [3 - 4\cos(2\varphi) + \cos(4\varphi)]/8.$

Back to (20), these are multiplied by $r^j = r^{p+q}$ for the intermediate Table 5 of Cartesian to spherical transformations.

Table 5. Examples of (20).

x	$= r\cos(\varphi).$
y	$= r\sin(\varphi).$
x^2	$= \frac{1}{2}r^2\cos(2\varphi) + \frac{1}{2}r^2.$
xy	$= \frac{1}{2}r^2\sin(2\varphi).$
y^2	$= -\frac{1}{2}r^2\cos(2\varphi) + \frac{1}{2}r^2.$
x^3	$= \frac{1}{4}r^3\cos(3\varphi) + \frac{3}{4}r^3\cos(\varphi).$
x^2y	$= \frac{1}{4}r^3\sin(3\varphi) + \frac{1}{4}r^3\sin(\varphi).$
xy^2	$= -\frac{1}{4}r^3\cos(3\varphi) + \frac{1}{4}r^3\cos(\varphi).$
y^3	$= -\frac{1}{4}r^3\sin(3\varphi) + \frac{3}{4}r^3\sin(\varphi).$
x^4	$= \frac{1}{8}r^4\cos(4\varphi) + \frac{1}{2}r^4\cos(2\varphi) + \frac{3}{8}r^4.$
x^3y	$= \frac{1}{8}r^4\sin(4\varphi) + \frac{1}{4}r^4\sin(2\varphi).$
x^2y^2	$= -\frac{1}{8}r^4\cos(4\varphi) + \frac{1}{8}r^4.$
xy^3	$= -\frac{1}{8}r^4\sin(4\varphi) + \frac{1}{4}r^4\sin(2\varphi).$
y^4	$= \frac{1}{8}r^4\cos(4\varphi) - \frac{1}{2}r^4\cos(2\varphi) + \frac{3}{8}r^4.$
x^5	$= \frac{1}{16}r^5\cos(5\varphi) + \frac{5}{16}r^5\cos(3\varphi) + \frac{5}{8}r^5\cos(\varphi).$
x^4y	$= \frac{1}{16}r^5\sin(5\varphi) + \frac{3}{16}r^5\sin(3\varphi) + \frac{1}{8}r^5\sin(\varphi).$
x^3y^2	$= -\frac{1}{16}r^5\cos(5\varphi) - \frac{1}{16}r^5\cos(3\varphi) + \frac{1}{8}r^5\cos(\varphi).$
x^2y^3	$= -\frac{1}{16}r^5\sin(5\varphi) + \frac{1}{16}r^5\sin(3\varphi) + \frac{1}{8}r^5\sin(\varphi).$
xy^4	$= \frac{1}{16}r^5\cos(5\varphi) - \frac{3}{16}r^5\cos(3\varphi) + \frac{1}{8}r^5\cos(\varphi).$
y^5	$= \frac{1}{16}r^5\sin(5\varphi) - \frac{5}{16}r^5\sin(3\varphi) + \frac{5}{8}r^5\sin(\varphi).$

Each factor r^j is expanded with (8) in a sum over R, selecting the line of the Table 2 associated with m to generate Table 6. For pure powers x^p, i.e., if the factor y is absent, the coefficients could also be taken from the column headed with two `p` in Conforti's (1983) Table 1.

2.4. Zernike to Cartesian

The reversal of the transformation of Chapter 2.3 splits each $r^j\cos(m\varphi)$ or $r^j\sin(m\varphi)$ in the

Table 6. Examples 2D Cartesian → Zernike.

x	$= R_1^1(r)\cos(\varphi).$
y	$= R_1^1(r)\sin(\varphi).$
x^2	$= \frac{1}{2}R_2^2(r)\cos(2\varphi) + \frac{1}{4}R_0^0(r) + \frac{1}{4}R_2^0(r).$
xy	$= \frac{1}{2}R_2^2(r)\sin(2\varphi).$
y^2	$= -\frac{1}{2}R_2^2(r)\cos(2\varphi) + \frac{1}{4}R_0^0(r) + \frac{1}{4}R_2^0(r).$
x^3	$= \frac{1}{4}R_3^3(r)\cos(3\varphi) + \frac{1}{2}\cos(\varphi)R_1^1(r) + \frac{1}{4}\cos(\varphi)R_3^1(r).$
x^2y	$= \frac{1}{4}R_3^3(r)\sin(3\varphi) + \frac{1}{6}\sin(\varphi)R_1^1(r) + \frac{1}{12}\sin(\varphi)R_3^1(r).$
xy^2	$= -\frac{1}{4}R_3^3(r)\cos(3\varphi) + \frac{1}{6}\cos(\varphi)R_1^1(r)$
	$+\frac{1}{12}\cos(\varphi)R_3^1(r).$
y^3	$= -\frac{1}{4}R_3^3(r)\sin(3\varphi) + \frac{1}{2}\sin(\varphi)R_1^1(r)$
	$+\frac{1}{4}\sin(\varphi)R_3^1(r).$
x^4	$= \frac{1}{8}R_4^4(r)\cos(4\varphi) + \frac{3}{8}\cos(2\varphi)R_2^2(r)$
	$+\frac{1}{8}\cos(2\varphi)R_4^2(r) + \frac{1}{8}R_0^0(r)$
	$+\frac{3}{16}R_2^0(r) + \frac{1}{16}R_4^0(r).$
x^3y	$= \frac{1}{8}R_4^4(r)\sin(4\varphi) + \frac{3}{16}\sin(2\varphi)R_2^2(r)$
	$+\frac{1}{16}\sin(2\varphi)R_4^2(r).$
x^2y^2	$= -\frac{1}{8}R_4^4(r)\cos(4\varphi) + \frac{1}{24}R_0^0(r) + \frac{1}{16}R_2^0(r) + \frac{1}{48}R_4^0(r).$
xy^3	$= -\frac{1}{8}R_4^4(r)\sin(4\varphi) + \frac{3}{16}\sin(2\varphi)R_2^2(r)$
	$+\frac{1}{16}\sin(2\varphi)R_4^2(r).$
y^4	$= \frac{1}{8}R_4^4(r)\cos(4\varphi) - \frac{3}{8}\cos(2\varphi)R_2^2(r)$
	$-\frac{1}{8}\cos(2\varphi)R_4^2(r) + \frac{1}{8}R_0^0(r)$
	$+\frac{3}{16}R_2^0(r) + \frac{1}{16}R_4^0(r).$
x^5	$= \frac{1}{16}R_5^5(r)\cos(5\varphi) + \frac{1}{4}\cos(3\varphi)R_3^3(r)$
	$+\frac{1}{16}\cos(3\varphi)R_5^3(r) + \frac{5}{16}\cos(\varphi)R_1^1(r)$
	$+\frac{1}{4}\cos(\varphi)R_3^1(r) + \frac{1}{16}\cos(\varphi)R_5^1(r).$
x^4y	$= \frac{1}{16}R_5^5(r)\sin(5\varphi) + \frac{3}{20}\sin(3\varphi)R_3^3(r)$
	$+\frac{3}{80}\sin(3\varphi)R_5^3(r) + \frac{1}{16}\sin(\varphi)R_1^1(r)$
	$+\frac{1}{20}\sin(\varphi)R_3^1(r) + \frac{1}{80}\sin(\varphi)R_5^1(r).$
x^3y^2	$= -\frac{1}{16}R_5^5(r)\cos(5\varphi) - \frac{1}{20}\cos(3\varphi)R_3^3(r)$
	$-\frac{1}{80}\cos(3\varphi)R_5^3(r) + \frac{1}{16}\cos(\varphi)R_1^1(r)$
	$+\frac{1}{20}\cos(\varphi)R_3^1(r) + \frac{1}{80}\cos(\varphi)R_5^1(r).$
x^2y^3	$= -\frac{1}{16}R_5^5(r)\sin(5\varphi) + \frac{1}{20}\sin(3\varphi)R_3^3(r)$
	$+\frac{1}{80}\sin(3\varphi)R_5^3(r) + \frac{1}{16}\sin(\varphi)R_1^1(r)$
	$+\frac{1}{20}\sin(\varphi)R_3^1(r) + \frac{1}{80}\sin(\varphi)R_5^1(r).$
xy^4	$= \frac{1}{16}R_5^5(r)\cos(5\varphi) - \frac{3}{20}\cos(3\varphi)R_3^3(r)$
	$-\frac{3}{80}\cos(3\varphi)R_5^3(r) + \frac{1}{16}\cos(\varphi)R_1^1(r)$
	$+\frac{1}{20}\cos(\varphi)R_3^1(r) + \frac{1}{80}\cos(\varphi)R_5^1(r).$
y^5	$= \frac{1}{16}R_5^5(r)\sin(5\varphi) - \frac{1}{4}\sin(3\varphi)R_3^3(r)$
	$-\frac{1}{16}\sin(3\varphi)R_5^3(r) + \frac{5}{16}\sin(\varphi)R_1^1(r)$
	$+\frac{1}{4}\sin(\varphi)R_3^1(r) + \frac{1}{16}\sin(\varphi)R_5^1(r).$

expansions of Chapter 2.1 into

$$r^{j-m}r^m\cos(m\varphi) = (x^2+y^2)^{(j-m)/2}\sum_{k=0,2,4,\ldots}^{m}(-1)^{\lfloor k/2\rfloor}\binom{m}{k}x^{m-k}y^k,$$

$$r^{j-m}r^m\sin(m\varphi) = (x^2+y^2)^{(j-m)/2}\sum_{k=1,3,5,\ldots}^{m}(-1)^{\lfloor k/2\rfloor}\binom{m}{k}x^{m-k}y^k,$$

which yields a sum over the bivariate Cartesian products after binomial expansion of $(x^2+y^2)^{(j-m)/2}$.

Table 7. Examples of (22) and (23).

$r\cos(\varphi)$	$= x.$
$r\sin(\varphi)$	$= y.$
r^2	$= x^2+y^2.$
$r^2\cos(2\varphi)$	$= x^2-y^2.$
$r^2\sin(2\varphi)$	$= 2xy.$
$r^3\cos(\varphi)$	$= x^3+xy^2.$
$r^3\sin(\varphi)$	$= yx^2+y^3.$
$r^3\cos(3\varphi)$	$= x^3-3xy^2.$
$r^3\sin(3\varphi)$	$= 3yx^2-y^3.$
r^4	$= x^4+2x^2y^2+y^4.$
$r^4\cos(2\varphi)$	$= x^4-y^4.$
$r^4\sin(2\varphi)$	$= 2x^3y+2xy^3.$
$r^4\cos(4\varphi)$	$= x^4-6x^2y^2+y^4.$
$r^4\sin(4\varphi)$	$= 4x^3y-4xy^3.$
$r^5\cos(\varphi)$	$= x^5+2x^3y^2+xy^4.$
$r^5\sin(\varphi)$	$= yx^4+2x^2y^3+y^5.$
$r^5\cos(3\varphi)$	$= x^5-2x^3y^2-3xy^4.$
$r^5\sin(3\varphi)$	$= 3yx^4+2x^2y^3-y^5.$
$r^5\cos(5\varphi)$	$= x^5-10x^3y^2+5xy^4.$
$r^5\sin(5\varphi)$	$= 5yx^4-10x^2y^3+y^5.$

The two cases are

$$r^j\cos(m\varphi)=\sum_{t=-\lfloor\frac{m}{2}\rfloor}^{(j-m)/2} x^{2t+m}y^{j-m-2t} \times \sum_{k=\max(0,-t)}^{\min(\frac{j-m}{2}-t,\lfloor\frac{m}{2}\rfloor)} (-1)^k\binom{m}{2k}\binom{\frac{j-m}{2}}{t+k}, \qquad (22)$$

and

$$r^j\sin(m\varphi)=\sum_{t=-\lfloor\frac{m-1}{2}\rfloor}^{(j-m)/2} x^{2t+m-1}y^{j-m-2t+1} \times \sum_{k=\max(0,-t)}^{\min(\frac{j-m}{2}-t,\lfloor\frac{m-1}{2}\rfloor)} (-1)^k\binom{m}{2k+1}\binom{\frac{j-m}{2}}{t+k}. \qquad (23)$$

The explicit tabulation for small j and small m is given in Table 7.

Linear superposition by inserting these into the right hand sides of (4) yields Table 8.

If m is odd, the transformation from $\cos(m\varphi)$ to $\sin(m\varphi)$ and vice versa is a straight exchange of x and y plus a multiplication by $(-1)^{\lfloor m/2\rfloor}$.

Table 8. Merger of Tables 1 and 7.

$R_0^0(r) = Z_1 = 1.$
$R_1^1(r)\cos(\varphi) = \frac{1}{2}Z_2 = x.$
$R_1^1(r)\sin(\varphi) = \frac{1}{2}Z_3 = y.$
$R_2^0(r) = \frac{1}{3}\sqrt{3}\,Z_4 = 2x^2+2y^2-1.$
$R_2^2(r)\cos(2\varphi) = \frac{1}{6}\sqrt{6}\,Z_6 = x^2-y^2.$
$R_2^2(r)\sin(2\varphi) = \frac{1}{6}\sqrt{6}\,Z_5 = 2xy.$
$R_3^1(r)\cos(\varphi) = \frac{1}{4}\sqrt{2}\,Z_8 = 3x^3+3xy^2-2x.$
$R_3^1(r)\sin(\varphi) = \frac{1}{4}\sqrt{2}\,Z_7 = 3x^2y+3y^3-2y.$
$R_3^3(r)\cos(3\varphi) = \frac{1}{4}\sqrt{2}\,Z_{10} = x^3-3xy^2.$
$R_3^3(r)\sin(3\varphi) = \frac{1}{4}\sqrt{2}\,Z_9 = 3x^2y-y^3.$
$R_4^0(r) = \frac{1}{5}\sqrt{5}\,Z_{11}$
$= 6x^4+12x^2y^2+6y^4-6x^2-6y^2+1.$
$R_4^2(r)\cos(2\varphi) = \frac{1}{10}\sqrt{10}\,Z_{12}$
$= 4x^4-4y^4-3x^2+3y^2.$
$R_4^2(r)\sin(2\varphi) = \frac{1}{10}\sqrt{10}\,Z_{13}$
$= 8x^3y+8xy^3-6xy.$
$R_4^4(r)\cos(4\varphi) = \frac{1}{10}\sqrt{10}\,Z_{14} = x^4-6x^2y^2+y^4.$
$R_4^4(r)\sin(4\varphi) = \frac{1}{10}\sqrt{10}\,Z_{15} = 4x^3y-4xy^3.$
$R_5^1(r)\cos(\varphi) = \frac{1}{6}\sqrt{3}\,Z_{16}$
$= 10x^5+20x^3y^2+10xy^4-12x^3-12xy^2+3x.$
$R_5^1(r)\sin(\varphi) = \frac{1}{6}\sqrt{3}\,Z_{17}$
$= 10x^4y+20x^2y^3+10y^5-12x^2y-12y^3+3y.$
$R_5^3(r)\cos(3\varphi) = \frac{1}{6}\sqrt{3}\,Z_{18}$
$= 5x^5-10x^3y^2-15xy^4-4x^3+12xy^2.$
$R_5^3(r)\sin(3\varphi) = \frac{1}{6}\sqrt{3}\,Z_{19}$
$= 15x^4y+10x^2y^3-5y^5-12x^2y+4y^3.$
$R_5^5(r)\cos(5\varphi) = \frac{1}{6}\sqrt{3}\,Z_{20} = x^5-10x^3y^2+5xy^4.$
$R_5^5(r)\sin(5\varphi) = \frac{1}{6}\sqrt{3}\,Z_{21} = 5x^4y-10x^2y^3+y^5.$

2.5. Product Expansion (Linearization Coefficients)

Products of Zernike functions are products $R_{n_1}^{m_1}R_{n_2}^{m_2}$ of the radial polynomials multiplied by products of azimuthal functions of essentially three different types,

$$\cos(m_1\varphi)\cos(m_2\varphi) = \frac{1}{2}\cos[(m_1-m_2)\varphi]+\frac{1}{2}\cos[(m_1+m_2)\varphi];$$
$$\sin(m_1\varphi)\cos(m_2\varphi) = \frac{1}{2}\sin[(m_1-m_2)\varphi]+\frac{1}{2}\sin[(m_1+m_2)\varphi];$$
$$\sin(m_1\varphi)\sin(m_2\varphi) = \frac{1}{2}\cos[(m_1-m_2)\varphi]-\frac{1}{2}\cos[(m_1+m_2)\varphi].$$

Since the azimuthal terms couple to $m_3=m_1\pm m_2$, most applications seek an expansion of the form

$$R_{n_1}^{m_1}(r)R_{n_2}^{m_2}(r)=\sum_{n_3=m_3}^{n_1+n_2} g_{n_1,m_1,n_2,m_2,n_3,m_3}R_{n_3}^{m_3}(r) \qquad (24)$$

given any of these two cases of m_3. The sum only covers even values of $n_1 + n_2 - n_3$. Equations (12) and (24) establish the sum rule

$$\sum_{n_3=(n_1+n_2) \bmod 2}^{n_1+n_2} g_{n_1,m_1,n_2,m_2,n_3,m_3} = 1 \qquad (25)$$

for the linearization coefficients.

The orthogonality (9) rewrites (24) as

$$\begin{aligned} &g_{n_1,m_1,n_2,m_2,n_3,m_3} \\ &= 2(n_3+1)\int_0^1 r R_{n_1}^{m_1}(r) R_{n_2}^{m_2}(r) R_{n_3}^{m_3}(r) dr. \qquad (26) \end{aligned}$$

Threefold use of (3) reduces the right hand side to a triple sum over elementary integrals over polynomials in r,

$$\begin{aligned} &g_{n_1,m_1,n_2,m_2,n_3,m_3} = 2(n_3+1) \\ &\times \sum_{s_1=0}^{-a_1}\sum_{s_2=0}^{-a_2}\sum_{s_3=0}^{-a_3} \frac{1}{n_1+n_2+n_3+2(1-s_1-s_2-s_3)} \\ &\times \prod_{t=1}^{3} (-1)^{s_t} \binom{n_t - s_t}{s_t}\binom{n_t - 2s_t}{-a_t - s_t}. \qquad (27) \end{aligned}$$

As an alternative, Bailey resummation (Slater 1966) of the polynomial product on the left hand side of (24) with $\sigma = s_1 + s_2$ and $1 + m_t = 1 - b_t + a_t$ for $t = 1$ or 2 gives

$$\begin{aligned} R_{n_1}^{m_1} R_{n_2}^{m_2} &= (-1)^{a_1+a_2}\binom{-b_1}{-a_1}\binom{-b_2}{-a_2} r^{m_1+m_2} \\ &\times \sum_{s_1=0}^{-a_1}\sum_{s_2=0}^{-a_2} \frac{(a_1)_{s_1}(1-b_1)_{s_1}}{(1+a_1-b_1)_{s_1} s_1!} \\ &\times \frac{(a_2)_{s_2}(1-b_2)_{s_2}}{(1+a_2-b_2)_{s_2} s_2!} r^{2(s_1+s_2)} \qquad (28) \\ &= (-1)^{a_1+a_2}\binom{-b_1}{-a_1}\binom{-b_2}{-a_2} r^{m_1+m_2} \\ &\times \sum_{\sigma=0}^{-a_1-a_2} r^{2\sigma} \frac{(a_2)_\sigma (1-b_2)_\sigma}{(1+a_2-b_2)_\sigma \sigma!} \\ &\times {}_4F_3(a_1, -\sigma, 1-b_1, -a_2+b_2-\sigma; \\ &\quad b_2-\sigma, 1+a_1-b_1, 1-a_2-\sigma; 1). \qquad (29) \end{aligned}$$

[The terminating Hypergeometric Function ${}_4F_3$ at the right hand side is "well poised" and has various other representations (Whipple 1926, Slater 1966).] This points at two more ways, besides (27), to compute g:

Table 9. Examples of (24).

$$\begin{aligned}
R_1^1(r)R_1^1(r) &= R_2^2(r) \\
&= \tfrac{1}{2}R_0^0(r) + \tfrac{1}{2}R_2^0(r). \\
R_1^1(r)R_2^0(r) &= \tfrac{1}{3}R_1^1(r) + \tfrac{2}{3}R_3^1(r). \\
R_1^1(r)R_2^2(r) &= R_3^3(r) \\
&= \tfrac{2}{3}R_1^1(r) + \tfrac{1}{3}R_3^1(r). \\
R_1^1(r)R_3^1(r) &= \tfrac{1}{4}R_2^2(r) + \tfrac{3}{4}R_4^2(r) \\
&= \tfrac{1}{2}R_2^0(r) + \tfrac{1}{2}R_4^0(r). \\
R_1^1(r)R_3^3(r) &= R_4^4(r) \\
&= \tfrac{3}{4}R_2^2(r) + \tfrac{1}{4}R_4^2(r). \\
R_2^0(r)R_2^0(r) &= \tfrac{1}{3}R_0^0(r) + \tfrac{2}{3}R_4^0(r). \\
R_2^0(r)R_2^2(r) &= \tfrac{1}{2}R_2^2(r) + \tfrac{1}{2}R_4^2(r). \\
R_2^2(r)R_2^2(r) &= R_4^4(r) \\
&= \tfrac{1}{3}R_0^0(r) + \tfrac{1}{2}R_2^0(r) + \tfrac{1}{6}R_4^0(r). \\
R_1^1(r)R_4^0(r) &= \tfrac{2}{5}R_3^1(r) + \tfrac{3}{5}R_5^1(r). \\
R_1^1(r)R_4^2(r) &= \tfrac{1}{5}R_3^3(r) + \tfrac{4}{5}R_5^3(r) \\
&= \tfrac{3}{5}R_3^1(r) + \tfrac{2}{5}R_5^1(r). \\
R_1^1(r)R_4^4(r) &= R_5^5(r) \\
&= \tfrac{4}{5}R_3^3(r) + \tfrac{1}{5}R_5^3(r). \\
R_2^0(r)R_3^1(r) &= \tfrac{1}{3}R_1^1(r) + \tfrac{1}{15}R_3^1(r) + \tfrac{3}{5}R_5^1(r). \\
R_2^0(r)R_3^3(r) &= \tfrac{3}{5}R_3^3(r) + \tfrac{2}{5}R_5^3(r). \\
R_2^2(r)R_3^1(r) &= \tfrac{2}{5}R_3^3(r) + \tfrac{3}{5}R_5^3(r) \\
&= \tfrac{1}{6}R_1^1(r) + \tfrac{8}{15}R_3^1(r) + \tfrac{3}{10}R_5^1(r). \\
R_2^2(r)R_3^3(r) &= R_5^5(r) \\
&= \tfrac{1}{2}R_1^1(r) + \tfrac{2}{5}R_3^1(r) + \tfrac{1}{10}R_5^1(r). \\
R_1^1(r)R_5^1(r) &= \tfrac{1}{3}R_4^2(r) + \tfrac{2}{3}R_6^2(r) \\
&= \tfrac{1}{2}R_4^0(r) + \tfrac{1}{2}R_6^0(r). \\
R_1^1(r)R_5^3(r) &= \tfrac{1}{6}R_4^4(r) + \tfrac{5}{6}R_6^4(r) \\
&= \tfrac{2}{3}R_4^2(r) + \tfrac{1}{3}R_6^2(r). \\
R_1^1(r)R_5^5(r) &= R_6^6(r) \\
&= \tfrac{5}{6}R_4^4(r) + \tfrac{1}{6}R_6^4(r). \\
R_2^0(r)R_4^0(r) &= \tfrac{2}{5}R_2^0(r) + \tfrac{3}{5}R_6^0(r). \\
R_2^0(r)R_4^2(r) &= \tfrac{3}{10}R_2^2(r) + \tfrac{1}{6}R_4^2(r) + \tfrac{8}{15}R_6^2(r). \\
R_2^0(r)R_4^4(r) &= \tfrac{2}{3}R_4^4(r) + \tfrac{1}{3}R_6^4(r). \\
R_2^2(r)R_4^0(r) &= \tfrac{1}{10}R_2^2(r) + \tfrac{1}{2}R_4^2(r) + \tfrac{2}{5}R_6^2(r). \\
R_2^2(r)R_4^2(r) &= \tfrac{1}{3}R_4^4(r) + \tfrac{2}{3}R_6^4(r) \\
&= \tfrac{3}{10}R_2^0(r) + \tfrac{1}{2}R_4^0(r) + \tfrac{1}{5}R_6^0(r). \\
R_2^2(r)R_4^4(r) &= R_6^6(r) \\
&= \tfrac{3}{5}R_2^2(r) + \tfrac{1}{3}R_4^2(r) + \tfrac{1}{15}R_6^2(r).
\end{aligned}$$

- Comparing coefficients of equal powers of r on both sides of (24), we obtain a linear system of equations. g with row index n_3 is the column vector of unknowns, the matrix has entries of the form

$$(-1)^{a_3}\binom{-b_3}{-a_3}\frac{(a_3)_{s_3}(1-b_3)_{s_3}}{(1+m_3)_{s_3} s_3!} \qquad (30)$$

  with column index n_3 and row index s_3, and the constant vector is given by the coefficients of (29) with row index $s_3 = \sigma_3 + (m_1 + m_2 - m_3)/2$.
- Substitution of the factors $r^{m_1+m_2+2\sigma}$ in (29) by (8) also converts the product to the format required by (24).

Omitting the trivial case of either n_1 or n_2 being zero, and showing only the cases $n_1 \le n_2$ (the others follow by swapping the two factors on the left hand sides) for $m_3 = m_1 \pm m_2$ generates Table 9. Recursive application expands triple, quadruple etc. products of Zernike functions.

3. 3D ZERNIKE FUNCTIONS

3.1. Radial Polynomials

The radial polynomials of the $D = 3$ Zernike functions (2) are (Mathar 2008, Novotni and Klein 2004)

$$R_n^{(l)}(r) = \frac{\sqrt{2n+3}}{2^{n-l}} \frac{(-1)^{(n-l)/2}}{\binom{n}{l}} r^l \times \sum_{s=0}^{(n-l)/2} \binom{n}{\frac{n-l}{2}-s} \binom{l+s}{l} \times \binom{l+1+n+2s}{n-l} (-r^2)^s \quad (31)$$

$$= \frac{\sqrt{2n+3}}{2^{n-l}} \frac{1}{\binom{n}{l}} \sum_{s=0}^{(n-l)/2} (-1)^s \binom{n}{s} \times \binom{l+\frac{n-l}{2}-s}{l} \binom{2n+1-2s}{n-l} r^{n-2s}, \quad (32)$$

defined for $0 \le l \le n$, even $n - l$, and normalized by design as

$$\int_0^1 r^2 R_n^{(l)}(r) R_{n'}^{(l)}(r) dr = \delta_{n,n'}. \quad (33)$$

A noteworthy special value is

$$R_n^{(l)}(1) = \sqrt{2n+3}, \quad (34)$$

which is derived from representations as terminating hypergeometric series or Jacobi Polynomials (Mathar 2008, Fields and Wimp 1961)

$$R_n^{(l)}(r) = \frac{\sqrt{2n+3}}{2^{n-l}} \frac{\binom{(l+n)/2}{l}\binom{1+2n}{n-l}}{\binom{n}{l}} r^n \times {}_2F_1\left(-\alpha, \alpha-n-\frac{1}{2}; -n-\frac{1}{2}; \frac{1}{r^2}\right) \quad (35)$$

$$= \sqrt{2n+3}\frac{(q)_\alpha}{\alpha!}(-1)^\alpha r^l \, {}_2F_1\left(-\alpha, q+\alpha; q; r^2\right), \quad (36)$$

where

$$\alpha \equiv (n-l)/2; \quad q \equiv l + 3/2. \quad (37)$$

To set them apart from the radial polynomials of Chapter 2, the upper index is enclosed in parentheses. A stylistic difference to the 2D case is that Noll (1976) maintained integer coefficients with the polynomials R_n^m by removing a factor $\sqrt{2n+2}$ from the R_n^m and a factor $\sqrt{\pi}$ from the azimuths, which resurface at places like (19) and (9). A recurrence is

$$(1+2n)(n+3+l)(n+2-l)\frac{R_{n+2}^{(l)}(r)}{\sqrt{2n+7}} + (5+2n)(l+1+n)(n-l)\frac{R_{n-2}^{(l)}(r)}{\sqrt{2n-1}} = (3+2n)\Big[(5+2n)(1+2n)r^2 - (2n^2+6n+2l+3+2l^2)\Big]\frac{R_n^{(l)}(r)}{\sqrt{2n+3}}. \quad (38)$$

Table 10. Examples of (32).

$R_0^{(0)}(r) = \sqrt{3}.$
$R_1^{(1)}(r) = \sqrt{5}\, r.$
$R_2^{(0)}(r) = \frac{1}{2}\sqrt{7}\,(-3+5r^2).$
$R_2^{(2)}(r) = \sqrt{7}\, r^2.$
$R_3^{(1)}(r) = \frac{3}{2} r(-5+7r^2).$
$R_3^{(3)}(r) = 3r^3.$
$R_4^{(0)}(r) = \frac{1}{8}\sqrt{11}\,(15-70r^2+63r^4).$
$R_4^{(2)}(r) = \frac{1}{2}\sqrt{11}\, r^2(-7+9r^2).$
$R_4^{(4)}(r) = \sqrt{11}\, r^4.$
$R_5^{(1)}(r) = \frac{1}{8}\sqrt{13}\, r(35-126r^2+99r^4).$
$R_5^{(3)}(r) = \frac{1}{2}\sqrt{13}\, r^3(-9+11r^2).$
$R_5^{(5)}(r) = \sqrt{13}\, r^5.$
$R_6^{(0)}(r) = \frac{1}{16}\sqrt{15}\,(-35+315r^2-693r^4+429r^6).$
$R_6^{(2)}(r) = \frac{1}{8}\sqrt{15}\, r^2(63-198r^2+143r^4).$
$R_6^{(4)}(r) = \frac{1}{2}\sqrt{15}\, r^4(-11+13r^2).$
$R_6^{(6)}(r) = \sqrt{15}\, r^6.$
$R_7^{(1)}(r) = \frac{1}{16}\sqrt{17}\, r(-105+693r^2-1287r^4 +715r^6).$
$R_7^{(3)}(r) = \frac{1}{8}\sqrt{17}\, r^3(99-286r^2+195r^4).$
$R_7^{(5)}(r) = \frac{1}{2}\sqrt{17}\, r^5(-13+15r^2).$
$R_7^{(7)}(r) = \sqrt{17}\, r^7.$
$R_8^{(0)}(r) = \frac{1}{128}\sqrt{19}\,(315-4620r^2+18018r^4 -25740r^6+12155r^8).$
$R_8^{(2)}(r) = \frac{1}{16}\sqrt{19}\, r^2(-231+1287r^2-2145r^4 +1105r^6).$
$R_8^{(4)}(r) = \frac{1}{8}\sqrt{19}\, r^4(143-390r^2+255r^4).$
$R_8^{(6)}(r) = \frac{1}{2}\sqrt{19}\, r^6(-15+17r^2).$
$R_8^{(8)}(r) = \sqrt{19}\, r^8.$

Inversion of (32) defines coefficients $f_{j,n,l}$,

$$r^j \equiv \sum_{n=l \bmod 2}^{j} f_{j,n,l} R_n^{(l)}(r), \quad j-l \text{ even}. \quad (39)$$

Table 11. Examples of (39).

$$
\begin{aligned}
r^2 &= \tfrac{1}{5}\sqrt{3}\,R_0^{(0)}(r) + \tfrac{2}{35}\sqrt{7}\,R_2^{(0)}(r)\\
r^3 &= \tfrac{1}{7}\sqrt{5}\,R_1^{(1)}(r) + \tfrac{2}{21}R_3^{(1)}(r).\\
r^4 &= \tfrac{1}{7}\sqrt{3}\,R_0^{(0)}(r) + \tfrac{4}{63}\sqrt{7}\,R_2^{(0)}(r)\\
&\quad + \tfrac{8}{693}\sqrt{11}\,R_4^{(0)}(r)\\
&= \tfrac{1}{9}\sqrt{7}\,R_2^{(2)}(r) + \tfrac{2}{99}\sqrt{11}\,R_4^{(2)}(r).\\
r^5 &= \tfrac{1}{9}\sqrt{5}\,R_1^{(1)}(r) + \tfrac{4}{33}R_3^{(1)}(r)\\
&\quad + \tfrac{8}{1287}\sqrt{13}\,R_5^{(1)}(r)\\
&= \tfrac{3}{11}R_3^{(3)}(r) + \tfrac{2}{143}\sqrt{13}\,R_5^{(3)}(r).\\
r^6 &= \tfrac{1}{9}\sqrt{3}\,R_0^{(0)}(r) + \tfrac{2}{33}\sqrt{7}\,R_2^{(0)}(r)\\
&\quad + \tfrac{8}{429}\sqrt{11}\,R_4^{(0)}(r) + \tfrac{16}{6435}\sqrt{15}\,R_6^{(0)}(r)\\
&= \tfrac{1}{11}\sqrt{7}\,R_2^{(2)}(r) + \tfrac{4}{143}\sqrt{11}\,R_4^{(2)}(r)\\
&\quad + \tfrac{8}{2145}\sqrt{15}\,R_6^{(2)}(r)\\
&= \tfrac{1}{13}\sqrt{11}\,R_4^{(4)}(r) + \tfrac{2}{195}\sqrt{15}\,R_6^{(4)}(r).\\
r^7 &= \tfrac{1}{11}\sqrt{5}\,R_1^{(1)}(r) + \tfrac{18}{143}R_3^{(1)}(r)\\
&\quad + \tfrac{8}{715}\sqrt{13}\,R_5^{(1)}(r) + \tfrac{16}{12155}\sqrt{17}\,R_7^{(1)}(r)\\
&= \tfrac{3}{13}R_3^{(3)}(r) + \tfrac{4}{195}\sqrt{13}\,R_5^{(3)}(r)\\
&\quad + \tfrac{8}{3315}\sqrt{17}\,R_7^{(3)}(r)\\
&= \tfrac{1}{15}\sqrt{13}\,R_5^{(5)}(r) + \tfrac{2}{255}\sqrt{17}\,R_7^{(5)}(r).\\
r^8 &= \tfrac{1}{11}\sqrt{3}\,R_0^{(0)}(r) + \tfrac{8}{143}\sqrt{7}\,R_2^{(0)}(r)\\
&\quad + \tfrac{16}{715}\sqrt{11}\,R_4^{(0)}(r) + \tfrac{64}{12155}\sqrt{15}\,R_6^{(0)}(r)\\
&\quad + \tfrac{128}{230945}\sqrt{19}\,R_8^{(0)}(r)\\
&= \tfrac{1}{13}\sqrt{7}\,R_2^{(2)}(r) + \tfrac{2}{65}\sqrt{11}\,R_4^{(2)}(r)\\
&\quad + \tfrac{8}{1105}\sqrt{15}\,R_6^{(2)}(r) + \tfrac{16}{20995}\sqrt{19}\,R_8^{(2)}(r)\\
&= \tfrac{1}{15}\sqrt{11}\,R_4^{(4)}(r) + \tfrac{4}{255}\sqrt{15}\,R_6^{(4)}(r)\\
&\quad + \tfrac{8}{4845}\sqrt{19}\,R_8^{(4)}(r)\\
&= \tfrac{1}{17}\sqrt{15}\,R_6^{(6)}(r) + \tfrac{2}{323}\sqrt{19}\,R_8^{(6)}(r).\\
r^9 &= \tfrac{1}{13}\sqrt{5}\,R_1^{(1)}(r) + \tfrac{8}{65}R_3^{(1)}(r)\\
&\quad + \tfrac{16}{1105}\sqrt{13}\,R_5^{(1)}(r) + \tfrac{64}{20995}\sqrt{17}\,R_7^{(1)}(r)\\
&\quad + \tfrac{128}{440895}\sqrt{21}\,R_9^{(1)}(r)\\
&= \tfrac{1}{5}R_3^{(3)}(r) + \tfrac{2}{85}\sqrt{13}\,R_5^{(3)}(r)\\
&\quad + \tfrac{8}{1615}\sqrt{17}\,R_7^{(3)}(r) + \tfrac{16}{33915}\sqrt{21}\,R_9^{(3)}(r)\\
&= \tfrac{1}{17}\sqrt{13}\,R_5^{(5)}(r) + \tfrac{4}{323}\sqrt{17}\,R_7^{(5)}(r)\\
&\quad + \tfrac{8}{6783}\sqrt{21}\,R_9^{(5)}(r)\\
&= \tfrac{1}{19}\sqrt{17}\,R_7^{(7)}(r) + \tfrac{2}{399}\sqrt{21}\,R_9^{(7)}(r).
\end{aligned}
$$

The projection technique equivalent to the 2D case based on the orthogonality (33), (36) and on Gradstein and Ryzhik's (1981) 7.512.2 yields

$$
\begin{aligned}
f_{j,n,l} &= \int_0^1 r^{j+2} R_n^{(l)}(r)dr\\
&= \sqrt{2n+3}\,\frac{\left(\frac{j-n}{2}+1\right)_\alpha}{(j+3+l)\left(\frac{j-l}{2}+q+1\right)_\alpha}. \qquad (40)
\end{aligned}
$$

Equation (39), evaluated at $r=1$, using (34) establishes the sum rule

$$\sum_{n=l \mod 2}^{j} \sqrt{2n+3} f_{j,n,l} = 1. \qquad (41)$$

Recurrences derived from (40) are

$$
\begin{aligned}
f_{j+2,n,l} &= \frac{(j+3+l)(j-l+2)}{(j-n+2)(j+n+5)} f_{j,n,l}; &(42)\\
f_{j,n+2,l} &= \frac{j-n}{j+5+n}\sqrt{\frac{2n+7}{2n+3}} f_{j,n,l}; &(43)\\
f_{j,n,l+2} &= \frac{j+3+l}{j-l} f_{j,n,l}. &(44)
\end{aligned}
$$

Examples of (39) for small powers j are gathered in Table 11. (Entries of the simple form $r^j = R_j^{(j)}/\sqrt{2j+3}$ are omitted.)

This expands r^j in a sum over $R_n^{(l)}$ at constant l. It might be regarded as solving (32) as a linear system of equations with a vector of unknowns from r^l to r^n,

$$
\begin{pmatrix} R_l^{(l)} \\ R_{l+2}^{(l)} \\ R_{l+4}^{(l)} \\ \vdots \\ R_n^{(l)} \end{pmatrix} =
\begin{pmatrix} \sqrt{2l+3} & 0 & 0 & 0 \\ \cdots & \ddots & 0 & 0 \\ & & \ddots & 0 \\ \cdots & \cdots & \cdots & \ddots \end{pmatrix} \cdot
\begin{pmatrix} r^l \\ r^{l+2} \\ r^{l+4} \\ \vdots \\ r^n \end{pmatrix}.
$$

The matrix is a lower triangular matrix displaying the coefficients of (32); the equation is easily solved via forward elimination, for example. A complementary expansion in a series of $R_n^{(l)}$ with constant n

$$r^j \equiv \sum_{l=j \mod 2}^{n} \hat{f}_{j,n,l} R_n^{(l)}(r), \quad j-n=0 \mod 2, \qquad (45)$$

lacks the orthogonality equivalent to (33), but again

Table 12. Examples of (45), $0 \le j \le 2$.

$$1\ /\sqrt{3} = \tfrac{1}{3}R_0^{(0)}.$$
$$1\ /\sqrt{7} = \tfrac{5}{21}R_2^{(2)} - \tfrac{2}{21}R_2^{(0)}.$$
$$1\ /\sqrt{11} = \tfrac{9}{55}R_4^{(4)} - \tfrac{4}{33}R_4^{(2)} + \tfrac{8}{165}R_4^{(0)}.$$
$$1\ /\sqrt{15} = \tfrac{13}{105}R_6^{(6)} - \tfrac{18}{175}R_6^{(4)} + \tfrac{8}{105}R_6^{(2)} - \tfrac{16}{525}R_6^{(0)}.$$
$$1\ /\sqrt{19} = \tfrac{17}{171}R_8^{(8)} - \tfrac{104}{1197}R_8^{(6)} + \tfrac{48}{665}R_8^{(4)} - \tfrac{64}{1197}R_8^{(2)} + \tfrac{128}{5985}R_8^{(0)}.$$
$$1\ /\sqrt{23} = \tfrac{21}{253}R_{10}^{(10)} - \tfrac{170}{2277}R_{10}^{(8)} + \tfrac{1040}{15939}R_{10}^{(6)} - \tfrac{96}{1771}R_{10}^{(4)} + \tfrac{640}{15939}R_{10}^{(2)} - \tfrac{256}{15939}R_{10}^{(0)}.$$
$$1\ /\sqrt{27} = \tfrac{25}{351}R_{12}^{(12)} - \tfrac{28}{429}R_{12}^{(10)} + \tfrac{680}{11583}R_{12}^{(8)} - \tfrac{320}{6237}R_{12}^{(6)} + \tfrac{128}{3003}R_{12}^{(4)} - \tfrac{2560}{81081}R_{12}^{(2)} + \tfrac{1024}{81081}R_{12}^{(0)}.$$
$$r\ /\sqrt{5} = \tfrac{1}{5}R_1^{(1)}.$$
$$r\ /3 = \tfrac{7}{45}R_3^{(3)} - \tfrac{2}{45}R_3^{(1)}.$$
$$r\ /\sqrt{13} = \tfrac{11}{91}R_5^{(5)} - \tfrac{4}{65}R_5^{(3)} + \tfrac{8}{455}R_5^{(1)}.$$
$$r\ /\sqrt{17} = \tfrac{5}{51}R_7^{(7)} - \tfrac{22}{357}R_7^{(5)} + \tfrac{8}{255}R_7^{(3)} - \tfrac{16}{1785}R_7^{(1)}.$$
$$r\ /\sqrt{21} = \tfrac{19}{231}R_9^{(9)} - \tfrac{40}{693}R_9^{(7)} + \tfrac{16}{441}R_9^{(5)} - \tfrac{64}{3465}R_9^{(3)} + \tfrac{128}{24255}R_9^{(1)}.$$
$$r\ /5 = \tfrac{23}{325}R_{11}^{(11)} - \tfrac{38}{715}R_{11}^{(9)} + \tfrac{16}{429}R_{11}^{(7)} - \tfrac{32}{1365}R_{11}^{(5)} + \tfrac{128}{10725}R_{11}^{(3)} - \tfrac{256}{75075}R_{11}^{(1)}.$$
$$r\ /\sqrt{29} = \tfrac{9}{145}R_{13}^{(13)} - \tfrac{92}{1885}R_{13}^{(11)} + \tfrac{152}{4147}R_{13}^{(9)} - \tfrac{320}{12441}R_{13}^{(7)} + \tfrac{128}{7917}R_{13}^{(5)} - \tfrac{512}{62205}R_{13}^{(3)} + \tfrac{1024}{435435}R_{13}^{(1)}.$$
$$r^2/\sqrt{7} = \tfrac{1}{7}R_2^{(2)}.$$
$$r^2/\sqrt{11} = \tfrac{9}{77}R_4^{(4)} - \tfrac{2}{77}R_4^{(2)}.$$
$$r^2/\sqrt{15} = \tfrac{13}{135}R_6^{(6)} - \tfrac{4}{105}R_6^{(4)} + \tfrac{8}{945}R_6^{(2)}.$$
$$r^2/\sqrt{19} = \tfrac{17}{209}R_8^{(8)} - \tfrac{26}{627}R_8^{(6)} + \tfrac{24}{1463}R_8^{(4)} - \tfrac{16}{4389}R_8^{(2)}.$$
$$r^2/\sqrt{23} = \tfrac{21}{299}R_{10}^{(10)} - \tfrac{136}{3289}R_{10}^{(8)} + \tfrac{16}{759}R_{10}^{(6)} - \tfrac{192}{23023}R_{10}^{(4)} + \tfrac{128}{69069}R_{10}^{(2)}.$$
$$r^2/\sqrt{27} = \tfrac{5}{81}R_{12}^{(12)} - \tfrac{14}{351}R_{12}^{(10)} + \tfrac{272}{11583}R_{12}^{(8)} - \tfrac{32}{2673}R_{12}^{(6)} + \tfrac{128}{27027}R_{12}^{(4)} - \tfrac{256}{243243}R_{12}^{(2)}.$$

it can be formulated as a linear system of equations,

$$\begin{pmatrix} R_n^{(l)} \\ R_n^{(l+2)} \\ R_n^{(l+4)} \\ \vdots \\ R_n^{(n)} \end{pmatrix} = \begin{pmatrix} \ddots & \cdots & \cdots & \cdots \\ 0 & \ddots & & \\ 0 & 0 & \ddots & \cdots \\ 0 & 0 & 0 & \sqrt{2n+3} \end{pmatrix} \cdot \begin{pmatrix} r^l \\ r^{l+2} \\ r^{l+4} \\ \vdots \\ r^n \end{pmatrix},$$

this time with an upper triangular matrix populated by the coefficients of (32), solvable with backward elimination. The sum rule according to (34) is

$$1 = \sqrt{2n+3}\sum_l \hat{f}_{j,n,l}. \tag{46}$$

Results are illustrated in Table 12. Unlike (39), which turns out to be helpful in Chapter 3.5, (45) is not used further below.

3.2. Basis in the Unit Sphere

Let the 3D Zernike functions

$$Z_{n,l}^{(m)} \equiv R_n^{(l)}(r)Y_l^{(m)}(\varphi,\theta); \tag{47}$$

be defined in ($0 \le \varphi \le 2\pi$; $0 \le \theta \le \pi$; $0 \le r \le 1$) with Edmonds' (1957) sign choice of the Spherical Harmonics (Messiah 1976)

$$Y_l^{(m)} = (-1)^m \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} P_l^m(\cos\theta)e^{im\varphi}, \tag{48}$$

($-l \le m \le l$). Complex conjugation, denoted by the star, yields (Edmonds 1957)

$$Y_l^{(m)*} = (-1)^m Y_l^{(-m)}. \tag{49}$$

The latitudes are spanned via Associated Legendre Functions (Edmonds 1957)

$$P_l^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_l(x), \quad m \ge 0, \tag{50}$$

differing by a factor $(-1)^m$ from other sign conventions (Abramowitz and Stegun 1972, Gradstein and Ryzhik 1981). Extension to negative upper indices is finally defined via

$$P_l^{-m}(x) = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x), \quad m > 0. \tag{51}$$

The 3D Zernike functions are products of Vector Harmonics $r^l Y_l^{(m)}$ by even polynomials in r of order $n-l$. They are ortho-normal over the unit sphere,

$$\int_{r<1} Z_{n,l}^{(m)}(r,\varphi,\theta)Z_{n',l'}^{(m')*}(r,\varphi,\theta)d^3r = \delta_{n,n'}\delta_{m,m'}\delta_{l,l'}. \tag{52}$$

This foundation in spherical coordinates spawns interest in transformation into (Section 3.3) or from (Section 3.4) Cartesian coordinates.

3.3. Zernike to Cartesian

The Spherical Harmonics are (Mathar 2002)

$$Y_l^{(m)}(\theta,\varphi) = (-1)^m \sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} 2^l \sin^m\theta \times \sum_{\nu=0}^{l-m} \left(\frac{m+\nu+1-l}{2}\right)_l \frac{1}{\nu!(l-m-\nu)!}\cos^\nu\theta \times \left(\sum_{t=0,2,4,\ldots}^{m} + i\sum_{t=1,3,5\ldots}^{m}\right)(-1)^{\lfloor t/2\rfloor} \times \binom{m}{t}\cos^{m-t}\varphi\sin^t\varphi, \quad (53)$$

where i is the imaginary unit. Rendered as Cartesian Coordinates

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\cos\varphi, \quad z = r\cos\theta,$$

the Vector Harmonics are the 3D version of Eqs. (22) and (23) (Mathar 2002),

$$r^l Y_l^{(m)}(\theta,\varphi) = \left(-\frac{1}{2}\right)^m \sqrt{\frac{2l+1}{4\pi}(l-m)!(l+m)!} \times \sum_{\sigma_1,\sigma_2\ge 0}^{\sigma_1+\sigma_2\le\frac{l-m}{2}} \frac{1}{\sigma_1!\sigma_2!}\left(-\frac{1}{4}\right)^{\sigma_1+\sigma_2} \times \frac{1}{(m+\sigma_1+\sigma_2)!(l-m-2(\sigma_1+\sigma_2))!} \times \left(\sum_{t=0,2,\ldots}^{m} + i\sum_{t=1,3,5\ldots}^{m}\right)(-1)^{\lfloor t/2\rfloor}\binom{m}{t} \times x^{m-t+2\sigma_1} y^{t+2\sigma_2} z^{l-m-2(\sigma_1+\sigma_2)}. \quad (54)$$

Sample outputs are shown in Table 13.

More general Cartesian representations of $r^n Y_l^{(m)}$, for even $n-l>0$, follow from there by multiplication with the trinomials

$$r^{n-l} = \sum_{\sigma_1,\sigma_2\ge 0}^{\sigma_1+\sigma_2\le(n-l)/2} \frac{[(n-l)/2]!}{\sigma_1!\sigma_2!(\frac{n-l}{2}-\sigma_1-\sigma_2)!} \times x^{2\sigma_1} y^{2\sigma_2} z^{n-l-2(\sigma_1+\sigma_2)}.$$

Gathering terms with the aid of Table 10 generates Cartesian expansions for $Z_{n,l}^{(m)}$ as in Table 14.

Table 13. Examples of (54).

$$\sqrt{\pi}\,Y_0^{(0)} = \tfrac{1}{2}.$$
$$\sqrt{\pi}\,rY_1^{(0)} = \tfrac{1}{2}\sqrt{3}\,z.$$
$$\sqrt{\pi}\,rY_1^{(1)} = -\tfrac{1}{4}\sqrt{6}\,x - \tfrac{1}{4}\sqrt{6}\,iy.$$
$$\sqrt{\pi}\,r^2Y_2^{(0)} = -\tfrac{1}{4}\sqrt{5}\,(-2z^2+y^2+x^2).$$
$$\sqrt{\pi}\,r^2Y_2^{(1)} = -\tfrac{1}{4}\sqrt{30}\,zx - \tfrac{1}{4}\sqrt{30}\,iyz.$$
$$\sqrt{\pi}\,r^2Y_2^{(2)} = -\tfrac{1}{8}\sqrt{30}\,(y^2-x^2) + \tfrac{1}{4}\sqrt{30}\,iyx.$$
$$\sqrt{\pi}\,r^3Y_3^{(0)} = -\tfrac{1}{4}\sqrt{7}\,(-2z^2+3y^2+3x^2)z.$$
$$\sqrt{\pi}\,r^3Y_3^{(1)} = \tfrac{1}{8}\sqrt{21}\,(x^2-4z^2+y^2)x + \tfrac{1}{8}\sqrt{21}\,iy(x^2-4z^2+y^2).$$
$$\sqrt{\pi}\,r^3Y_3^{(2)} = -\tfrac{1}{8}\sqrt{210}\,(y^2-x^2)z + \tfrac{1}{4}\sqrt{210}\,ixyz.$$
$$\sqrt{\pi}\,r^3Y_3^{(3)} = \tfrac{1}{8}\sqrt{35}\,(-x^2+3y^2)x + \tfrac{1}{8}\sqrt{35}\,iy(-3x^2+y^2).$$
$$\sqrt{\pi}\,r^4Y_4^{(0)} = \tfrac{3}{2}z^4 - \tfrac{9}{2}y^2z^2 + \tfrac{9}{16}y^4 - \tfrac{9}{2}z^2x^2 + \tfrac{9}{8}y^2x^2 + \tfrac{9}{16}x^4.$$
$$\sqrt{\pi}\,r^4Y_4^{(1)} = \tfrac{3}{8}\sqrt{5}\,(-4z^2+3x^2+3y^2)zx + \tfrac{3}{8}\sqrt{5}\,i(-4z^2+3x^2+3y^2)yz.$$
$$\sqrt{\pi}\,r^4Y_4^{(2)} = \tfrac{3}{16}\sqrt{10}\,(y^2-6z^2+x^2)(y^2-x^2) - \tfrac{3}{8}\sqrt{10}\,iy(y^2-6z^2+x^2)x.$$
$$\sqrt{\pi}\,r^4Y_4^{(3)} = \tfrac{3}{8}\sqrt{35}\,(-x^2+3y^2)zx + \tfrac{3}{8}\sqrt{35}\,iy(-3x^2+y^2)z.$$
$$\sqrt{\pi}\,r^4Y_4^{(4)} = \tfrac{3}{32}\sqrt{70}\,(-x^2-2yx+y^2) \times(-x^2+2yx+y^2) - \tfrac{3}{8}\sqrt{70}\,i(y^2-x^2)yx.$$
$$\sqrt{\pi}\,r^5Y_5^{(0)} = \tfrac{1}{16}\sqrt{11}\,(8z^4-40z^2x^2-40y^2z^2 + 30y^2x^2+15x^4+15y^4)z.$$
$$\sqrt{\pi}\,r^5Y_5^{(1)} = -\tfrac{1}{32}\sqrt{330}\,(x^4-12z^2x^2+2y^2x^2 + 8z^4-12y^2z^2+y^4)x - \tfrac{1}{32}\sqrt{330}\,iy(x^4-12z^2x^2+2y^2x^2 + 8z^4-12y^2z^2+y^4).$$
$$\sqrt{\pi}\,r^5Y_5^{(2)} = \tfrac{1}{16}\sqrt{2310}\,(-2z^2+y^2+x^2)(y^2-x^2)z - \tfrac{1}{8}\sqrt{2310}\,iy(-2z^2+y^2+x^2)zx.$$
$$\sqrt{\pi}\,r^5Y_5^{(3)} = -\tfrac{1}{32}\sqrt{385}\,(y^2+x^2-8z^2)(-x^2+3y^2)x - \tfrac{1}{32}\sqrt{385}\,i(y^2+x^2-8z^2)(-3x^2+y^2)y.$$
$$\sqrt{\pi}\,r^5Y_5^{(4)} = \tfrac{3}{32}\sqrt{770}\,(-x^2-2yx+y^2) \times(-x^2+2yx+y^2)z - \tfrac{3}{8}\sqrt{770}\,i(y^2-x^2)yzx.$$
$$\sqrt{\pi}\,r^5Y_5^{(5)} = -\tfrac{3}{32}\sqrt{77}\,(x^4-10y^2x^2+5y^4)x - \tfrac{3}{32}\sqrt{77}\,iy(5x^4-10y^2x^2+y^4).$$

Terms with negative upper index m of Z have not been listed and follow from (49): If m is even, the imaginary part of Z changes sign; if m is odd, the real part changes sign.

Table 14. Cartesian representations of (47).

$Z_{0,0}^{(0)} = \frac{1}{2}\sqrt{3/\pi}.$
$Z_{1,1}^{(0)} = \frac{1}{2}\sqrt{15/\pi}\,z.$
$Z_{1,1}^{(1)} = -\frac{1}{4}\sqrt{30/\pi}\,x - \frac{1}{4}\sqrt{30/\pi}\,iy.$
$Z_{2,0}^{(0)} = \frac{1}{4}\sqrt{7/\pi}\,(-3+5x^2+5y^2+5z^2).$
$Z_{2,2}^{(0)} = -\frac{1}{4}\sqrt{35/\pi}\,(-2z^2+y^2+x^2).$
$Z_{2,2}^{(1)} = -\frac{1}{4}\sqrt{210/\pi}\,xz - \frac{1}{4}\sqrt{210/\pi}\,iyz.$
$Z_{2,2}^{(2)} = -\frac{1}{8}\sqrt{210/\pi}\,(y^2-x^2) + \frac{1}{4}\sqrt{210/\pi}\,iyx.$
$Z_{3,1}^{(0)} = \frac{3}{4}\sqrt{3/\pi}\,z(-5+7x^2+7y^2+7z^2).$
$Z_{3,1}^{(1)} = -\frac{3}{8}\sqrt{6/\pi}\,x(-5+7x^2+7y^2+7z^2)$ $-\frac{3}{8}\sqrt{6/\pi}\,iy(-5+7x^2+7y^2+7z^2).$
$Z_{3,3}^{(0)} = -\frac{3}{4}\sqrt{7/\pi}\,z(-2z^2+3y^2+3x^2).$
$Z_{3,3}^{(1)} = \frac{3}{8}\sqrt{21/\pi}\,x(x^2-4z^2+y^2)$ $+\frac{3}{8}\sqrt{21/\pi}\,iy(x^2-4z^2+y^2).$
$Z_{3,3}^{(2)} = -\frac{3}{8}\sqrt{210/\pi}\,z(y^2-x^2) + \frac{3}{4}\sqrt{210/\pi}\,ixyz.$
$Z_{3,3}^{(3)} = \frac{3}{8}\sqrt{35/\pi}\,x(-x^2+3y^2)$ $+\frac{3}{8}\sqrt{35/\pi}\,iy(-3x^2+y^2).$
$Z_{4,0}^{(0)} = \frac{1}{16}\sqrt{11/\pi}\,(15-70x^2-70y^2-70z^2$ $+63x^4+126y^2x^2+126z^2x^2+63y^4$ $+126y^2z^2+63z^4).$
$Z_{4,2}^{(0)} = -\frac{1}{8}\sqrt{55/\pi}\,(-2z^2+y^2+x^2)$ $\times(-7+9x^2+9y^2+9z^2).$
$Z_{4,2}^{(1)} = -\frac{1}{8}\sqrt{330/\pi}\,xz(-7+9x^2+9y^2+9z^2)$ $-\frac{1}{8}\sqrt{330/\pi}\,iyz(-7+9x^2+9y^2+9z^2).$
$Z_{4,2}^{(2)} = -\frac{1}{16}\sqrt{330/\pi}\,(y^2-x^2)$ $\times(-7+9x^2+9y^2+9z^2)$ $+\frac{1}{8}\sqrt{330/\pi}\,iyx(-7+9x^2+9y^2+9z^2).$
$Z_{4,4}^{(0)} = \frac{3}{16}\sqrt{11/\pi}\,(8z^4-24y^2z^2+3y^4$ $-24z^2x^2+6y^2x^2+3x^4).$
$Z_{4,4}^{(1)} = \frac{3}{8}\sqrt{55/\pi}\,xz(-4z^2+3y^2+3x^2)$ $+\frac{3}{8}\sqrt{55/\pi}\,iyz(-4z^2+3y^2+3x^2).$
$Z_{4,4}^{(2)} = \frac{3}{16}\sqrt{110/\pi}\,(y^2-x^2)(y^2-6z^2+x^2)$ $-\frac{3}{8}\sqrt{110/\pi}\,iyx(y^2-6z^2+x^2).$
$Z_{4,4}^{(3)} = \frac{3}{8}\sqrt{385/\pi}\,xz(-x^2+3y^2)$ $+\frac{3}{8}\sqrt{385/\pi}\,iyz(-3x^2+y^2).$
$Z_{4,4}^{(4)} = \frac{3}{32}\sqrt{770/\pi}\,(-x^2+2yx+y^2)$ $\times(-x^2-2yx+y^2)$ $-\frac{3}{8}\sqrt{770/\pi}\,iyx(y^2-x^2).$

3.4. Cartesian to Zernike

The inverse problem to the one of Section 3.3 is finding the u-coefficients in the ansatz

$$x^p y^q z^t = \sum_{\substack{-p-q\le m\le p+q \\ l\le n\le p+q+t}} u_{p,q,t,n,l,m} R_n^{(l)}(r) Y_l^{(m)}(\theta,\varphi), \tag{55}$$

given p, q and t. We use (52) to construct

$$u_{p,q,t,n,l,m} = \int_{r<1} r^{p+q+t}\sin^{p+q}\theta\cos^t\theta \times \cos^p\varphi\sin^q\varphi R_n^{(l)}(r)Y_l^{(m)*}(\theta,\varphi)d^3r$$

by projection onto each individual $Z_{n,l}^{(m)}$. Section 3.3 of my earlier work (Mathar 2002) factorizes this triple integral as

$$u_{p,q,t,n,l,m} = (-1)^m\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}} \times I_r(p+q+t,n,l)I_\theta(p+q,t,l,m)I_\varphi(p,q,m)^*,$$

with two factors defined as

$$I_r(j,n,l) = \int_0^1 r^{j+2}R_n^{(l)}(r)dr = \begin{cases} 0, & j-l\,\text{odd or}\,n-l\,\text{odd}; \\ f_{j,n,l}, & j+l\,\text{even}. \end{cases}$$

and

$$I_\varphi(p,q,m) = \int_0^{2\pi} e^{im\varphi}\cos^p\varphi\sin^q\varphi d\varphi$$

$$= \begin{cases} 0, p+q-m\,\text{odd}; \\ \frac{\pi}{2^{p+q-1}} i^q \sum_{\sigma=\max(0,\frac{p-q-m}{2})}^{\min(p,\frac{p+q-m}{2})} \binom{p}{\sigma} \\ \quad\times\binom{q}{\frac{p+q-m}{2}-\sigma}(-1)^{\sigma-(p+q-m)/2}, \\ \quad p+q-m\,\text{even}. \end{cases}$$

An obvious recurrence is

$$I_\varphi(p,q+2,m) = I_\varphi(p,q,m) - I_\varphi(p+2,q,m). \tag{56}$$

The third factor I_θ is only evaluated for even $p+q-m$—otherwise I_φ equals zero which turns u to zero and I_θ is not of interest—

$$I_\theta(k,t,l,m) = \int_0^\pi \sin^{k+1}\theta\cos^t\theta P_l^m(\cos\theta)d\theta$$

$$= \begin{cases} 0, \quad l-m+t\,\text{odd}; \\ \frac{2^{l+1}(-1)^{(m-|m|)/2}}{(l-m)!}\sum_{\nu=0}^{(l-|m|)/2}\frac{\left(\frac{1}{2}-\nu\right)_l}{1+t+l-|m|-2\nu} \\ \quad\times\frac{\binom{l-|m|}{2\nu}}{\binom{(1+t+l+k)/2-\nu}{(k+|m|)/2}}, \quad l-m+t\,\text{even}. \end{cases}$$

This computation of u generates Table 15.

Table 15. Examples of (55).

$$x/\sqrt{\pi} = \tfrac{1}{15}\sqrt{30}\,Z^{(-1)}_{1,1} - \tfrac{1}{15}\sqrt{30}\,Z^{(1)}_{1,1}.$$
$$y/\sqrt{\pi} = \tfrac{i}{15}\sqrt{30}\,Z^{(-1)}_{1,1} + \tfrac{i}{15}\sqrt{30}\,Z^{(1)}_{1,1}.$$
$$z/\sqrt{\pi} = \tfrac{2}{15}\sqrt{15}\,Z^{(0)}_{1,1}.$$
$$x^2/\sqrt{\pi} = \tfrac{2}{15}\sqrt{3}\,Z^{(0)}_{0,0} + \tfrac{4}{105}\sqrt{7}\,Z^{(0)}_{2,0} + \tfrac{1}{105}\sqrt{210}\,Z^{(-2)}_{2,2} - \tfrac{2}{105}\sqrt{35}\,Z^{(0)}_{2,2} + \tfrac{1}{105}\sqrt{210}\,Z^{(2)}_{2,2}.$$
$$xy/\sqrt{\pi} = \tfrac{i}{105}\sqrt{210}\,Z^{(-2)}_{2,2} - \tfrac{i}{105}\sqrt{210}\,Z^{(2)}_{2,2}.$$
$$xz/\sqrt{\pi} = \tfrac{1}{105}\sqrt{210}\,Z^{(-1)}_{2,2} - \tfrac{1}{105}\sqrt{210}\,Z^{(1)}_{2,2}.$$
$$y^2/\sqrt{\pi} = \tfrac{2}{15}\sqrt{3}\,Z^{(0)}_{0,0} + \tfrac{4}{105}\sqrt{7}\,Z^{(0)}_{2,0} - \tfrac{1}{105}\sqrt{210}\,Z^{(-2)}_{2,2} - \tfrac{2}{105}\sqrt{35}\,Z^{(0)}_{2,2} - \tfrac{1}{105}\sqrt{210}\,Z^{(2)}_{2,2}.$$
$$yz/\sqrt{\pi} = \tfrac{i}{105}\sqrt{210}\,Z^{(-1)}_{2,2} + \tfrac{i}{105}\sqrt{210}\,Z^{(1)}_{2,2}.$$
$$z^2/\sqrt{\pi} = \tfrac{2}{15}\sqrt{3}\,Z^{(0)}_{0,0} + \tfrac{4}{105}\sqrt{7}\,Z^{(0)}_{2,0} + \tfrac{4}{105}\sqrt{35}\,Z^{(0)}_{2,2}.$$
$$x^3/\sqrt{\pi} = \tfrac{1}{35}\sqrt{30}\,Z^{(-1)}_{1,1} - \tfrac{1}{35}\sqrt{30}\,Z^{(1)}_{1,1} + \tfrac{2}{105}\sqrt{6}\,Z^{(-1)}_{3,1} - \tfrac{2}{105}\sqrt{6}\,Z^{(1)}_{3,1} + \tfrac{1}{105}\sqrt{35}\,Z^{(-3)}_{3,3} - \tfrac{1}{105}\sqrt{21}\,Z^{(-1)}_{3,3} + \tfrac{1}{105}\sqrt{21}\,Z^{(1)}_{3,3} - \tfrac{1}{105}\sqrt{35}\,Z^{(3)}_{3,3}.$$
$$x^2y/\sqrt{\pi} = \tfrac{i}{105}\sqrt{30}\,Z^{(-1)}_{1,1} + \tfrac{i}{105}\sqrt{30}\,Z^{(1)}_{1,1} + \tfrac{2i}{315}\sqrt{6}\,Z^{(-1)}_{3,1} + \tfrac{2i}{315}\sqrt{6}\,Z^{(1)}_{3,1} + \tfrac{i}{105}\sqrt{35}\,Z^{(-3)}_{3,3} - \tfrac{i}{315}\sqrt{21}\,Z^{(-1)}_{3,3} - \tfrac{i}{315}\sqrt{21}\,Z^{(1)}_{3,3} + \tfrac{i}{105}\sqrt{35}\,Z^{(3)}_{3,3}.$$
$$x^2z/\sqrt{\pi} = \tfrac{2}{105}\sqrt{15}\,Z^{(0)}_{1,1} + \tfrac{4}{315}\sqrt{3}\,Z^{(0)}_{3,1} + \tfrac{1}{315}\sqrt{210}\,Z^{(-2)}_{3,3} - \tfrac{2}{105}\sqrt{7}\,Z^{(0)}_{3,3} + \tfrac{1}{315}\sqrt{210}\,Z^{(2)}_{3,3}.$$
$$xy^2/\sqrt{\pi} = \tfrac{1}{105}\sqrt{30}\,Z^{(-1)}_{1,1} - \tfrac{1}{105}\sqrt{30}\,Z^{(1)}_{1,1} + \tfrac{2}{315}\sqrt{6}\,Z^{(-1)}_{3,1} - \tfrac{2}{315}\sqrt{6}\,Z^{(1)}_{3,1} - \tfrac{1}{105}\sqrt{35}\,Z^{(-3)}_{3,3} - \tfrac{1}{315}\sqrt{21}\,Z^{(-1)}_{3,3} + \tfrac{1}{315}\sqrt{21}\,Z^{(1)}_{3,3} + \tfrac{1}{105}\sqrt{35}\,Z^{(3)}_{3,3}.$$
$$xyz/\sqrt{\pi} = \tfrac{i}{315}\sqrt{210}\,Z^{(-2)}_{3,3} - \tfrac{i}{315}\sqrt{210}\,Z^{(2)}_{3,3}.$$
$$xz^2/\sqrt{\pi} = \tfrac{1}{105}\sqrt{30}\,Z^{(-1)}_{1,1} - \tfrac{1}{105}\sqrt{30}\,Z^{(1)}_{1,1} + \tfrac{2}{315}\sqrt{6}\,Z^{(-1)}_{3,1} - \tfrac{2}{315}\sqrt{6}\,Z^{(1)}_{3,1} + \tfrac{4}{315}\sqrt{21}\,Z^{(-1)}_{3,3} - \tfrac{4}{315}\sqrt{21}\,Z^{(1)}_{3,3}.$$
$$y^3/\sqrt{\pi} = \tfrac{i}{35}\sqrt{30}\,Z^{(-1)}_{1,1} + \tfrac{i}{35}\sqrt{30}\,Z^{(1)}_{1,1} + \tfrac{2i}{105}\sqrt{6}\,Z^{(-1)}_{3,1} + \tfrac{2i}{105}\sqrt{6}\,Z^{(1)}_{3,1} - \tfrac{i}{105}\sqrt{35}\,Z^{(-3)}_{3,3} - \tfrac{i}{105}\sqrt{21}\,Z^{(-1)}_{3,3} - \tfrac{i}{105}\sqrt{21}\,Z^{(1)}_{3,3} - \tfrac{i}{105}\sqrt{35}\,Z^{(3)}_{3,3}.$$
$$y^2z/\sqrt{\pi} = \tfrac{2}{105}\sqrt{15}\,Z^{(0)}_{1,1} + \tfrac{4}{315}\sqrt{3}\,Z^{(0)}_{3,1} - \tfrac{1}{315}\sqrt{210}\,Z^{(-2)}_{3,3} - \tfrac{2}{105}\sqrt{7}\,Z^{(0)}_{3,3} - \tfrac{1}{315}\sqrt{210}\,Z^{(2)}_{3,3}.$$
$$yz^2/\sqrt{\pi} = \tfrac{i}{105}\sqrt{30}\,Z^{(-1)}_{1,1} + \tfrac{i}{105}\sqrt{30}\,Z^{(1)}_{1,1} + \tfrac{2i}{315}\sqrt{6}\,Z^{(-1)}_{3,1} + \tfrac{2i}{315}\sqrt{6}\,Z^{(1)}_{3,1} + \tfrac{4i}{315}\sqrt{21}\,Z^{(-1)}_{3,3} + \tfrac{4i}{315}\sqrt{21}\,Z^{(1)}_{3,3}.$$

3.5. Product Expansion (Linearization Coefficients)

The product of the angular variables are the established expansions with Wigner $3j$ coefficients (Edmonds 1957, Sébilleau 1998)

$$Y^{(m_1)}_{l_1} Y^{(m_2)}_{l_2} = \sum_{l=|l_1-l_2|}^{l_1+l_2} \sum_{m=-l}^{l} \sqrt{\frac{(2l_1+1)(2l_2+1)(2l+1)}{4\pi}} \times \begin{pmatrix} l_1 & l_2 & l \\ m_1 & m_2 & m \end{pmatrix} \begin{pmatrix} l_1 & l_2 & l \\ 0 & 0 & 0 \end{pmatrix} Y^{(m)*}_{l}, \quad (57)$$

where the sum over m is nonzero only at $m_1+m_2+m=0$. The relation to Clebsch-Gordan coefficients (Rasch 2003)

Table 16. Examples of (57).

$$\sqrt{\pi}\,Y^{(0)}_0 Y^{(0)}_0 = \sqrt{2}\,Y^{(0)}_0.$$
$$\sqrt{\pi}\,Y^{(-1)}_1 Y^{(0)}_0 = \tfrac{1}{3}\sqrt{2}\,Y^{(-1)}_1.$$
$$\sqrt{\pi}\,Y^{(-1)}_1 Y^{(-1)}_1 = \tfrac{1}{30}\sqrt{6}\,Y^{(-2)}_2.$$
$$\sqrt{\pi}\,Y^{(0)}_1 Y^{(0)}_0 = \tfrac{2}{3}\sqrt{2}\,Y^{(0)}_1.$$
$$\sqrt{\pi}\,Y^{(0)}_1 Y^{(-1)}_1 = \tfrac{1}{15}\sqrt{6}\,Y^{(-1)}_2.$$
$$\sqrt{\pi}\,Y^{(0)}_1 Y^{(0)}_1 = \tfrac{2}{3}\sqrt{2}\,Y^{(0)}_0 + \tfrac{4}{15}\sqrt{6}\,Y^{(0)}_2.$$
$$\sqrt{\pi}\,Y^{(1)}_1 Y^{(0)}_0 = \tfrac{1}{3}\sqrt{2}\,Y^{(1)}_1.$$
$$\sqrt{\pi}\,Y^{(1)}_1 Y^{(-1)}_1 = -\tfrac{1}{3}\sqrt{2}\,Y^{(0)}_0 + \tfrac{1}{30}\sqrt{6}\,Y^{(0)}_2.$$
$$\sqrt{\pi}\,Y^{(1)}_1 Y^{(0)}_1 = \tfrac{1}{15}\sqrt{6}\,Y^{(1)}_2.$$
$$\sqrt{\pi}\,Y^{(1)}_1 Y^{(1)}_1 = \tfrac{1}{30}\sqrt{6}\,Y^{(2)}_2.$$
$$\sqrt{\pi}\,Y^{(-2)}_2 Y^{(0)}_0 = \tfrac{1}{40}\sqrt{2}\,Y^{(-2)}_2.$$
$$\sqrt{\pi}\,Y^{(-2)}_2 Y^{(-1)}_1 = \tfrac{1}{420}\sqrt{3}\,Y^{(-3)}_3.$$
$$\sqrt{\pi}\,Y^{(-2)}_2 Y^{(0)}_1 = \tfrac{1}{210}\sqrt{3}\,Y^{(-2)}_3.$$
$$\sqrt{\pi}\,Y^{(-2)}_2 Y^{(1)}_1 = -\tfrac{1}{30}\sqrt{6}\,Y^{(-1)}_1 + \tfrac{1}{420}\sqrt{3}\,Y^{(-1)}_3.$$
$$\sqrt{\pi}\,Y^{(-2)}_2 Y^{(-2)}_2 = \tfrac{1}{20160}\sqrt{10}\,Y^{(-4)}_4.$$
$$\sqrt{\pi}\,Y^{(-1)}_2 Y^{(0)}_0 = \tfrac{1}{10}\sqrt{2}\,Y^{(-1)}_2.$$
$$\sqrt{\pi}\,Y^{(-1)}_2 Y^{(-1)}_1 = \tfrac{1}{105}\sqrt{3}\,Y^{(-2)}_3.$$
$$\sqrt{\pi}\,Y^{(-1)}_2 Y^{(0)}_1 = \tfrac{1}{15}\sqrt{6}\,Y^{(-1)}_1 + \tfrac{2}{105}\sqrt{3}\,Y^{(-1)}_3.$$
$$\sqrt{\pi}\,Y^{(-1)}_2 Y^{(1)}_1 = -\tfrac{1}{15}\sqrt{6}\,Y^{(0)}_1 + \tfrac{1}{105}\sqrt{3}\,Y^{(0)}_3.$$
$$\sqrt{\pi}\,Y^{(-1)}_2 Y^{(-2)}_2 = \tfrac{1}{5040}\sqrt{10}\,Y^{(-3)}_4.$$
$$\sqrt{\pi}\,Y^{(-1)}_2 Y^{(-1)}_2 = \tfrac{1}{140}\sqrt{6}\,Y^{(-2)}_2 + \tfrac{1}{1260}\sqrt{10}\,Y^{(-2)}_4.$$
$$\sqrt{\pi}\,Y^{(0)}_2 Y^{(0)}_0 = \tfrac{3}{5}\sqrt{2}\,Y^{(0)}_2.$$
$$\sqrt{\pi}\,Y^{(0)}_2 Y^{(-1)}_1 = -\tfrac{1}{30}\sqrt{6}\,Y^{(-1)}_1 + \tfrac{1}{70}\sqrt{3}\,Y^{(-1)}_3.$$
$$\sqrt{\pi}\,Y^{(0)}_2 Y^{(0)}_1 = \tfrac{4}{15}\sqrt{6}\,Y^{(0)}_1 + \tfrac{12}{35}\sqrt{3}\,Y^{(0)}_3.$$
$$\sqrt{\pi}\,Y^{(0)}_2 Y^{(1)}_1 = -\tfrac{1}{30}\sqrt{6}\,Y^{(1)}_1 + \tfrac{1}{70}\sqrt{3}\,Y^{(1)}_3.$$
$$\sqrt{\pi}\,Y^{(0)}_2 Y^{(-2)}_2 = -\tfrac{1}{280}\sqrt{6}\,Y^{(-2)}_2 + \tfrac{1}{3360}\sqrt{10}\,Y^{(-2)}_4.$$
$$\sqrt{\pi}\,Y^{(0)}_2 Y^{(-1)}_2 = \tfrac{1}{140}\sqrt{6}\,Y^{(-1)}_2 + \tfrac{1}{840}\sqrt{10}\,Y^{(-1)}_4.$$
$$\sqrt{\pi}\,Y^{(0)}_2 Y^{(0)}_2 = \tfrac{3}{5}\sqrt{2}\,Y^{(0)}_0 + \tfrac{6}{35}\sqrt{6}\,Y^{(0)}_2 + \tfrac{6}{35}\sqrt{10}\,Y^{(0)}_4.$$

$$\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} = (-1)^{j_1-j_2-m_3}\frac{1}{\sqrt{2j_3+1}} \times (j_1 m_1 j_2 m_2 \mid j_1 j_2 j_3 - m_3)$$

opens a route to computation via standard formulas (Edmonds 1957, Fano 1959, Abramowitz and Stegun 1972, Ulfbeck 2007). The familiar Table 16 results in conjunction with (49).

Table 17. Examples of (58). All arguments are (r).

$$\begin{aligned}
R_1^{(1)} R_1^{(1)} &= \sqrt{3}\, R_0^{(0)} + \tfrac{2}{7}\sqrt{7}\, R_2^{(0)} \\
&= \tfrac{5}{7}\sqrt{7}\, R_2^{(2)}. \\
R_1^{(1)} R_2^{(0)} &= \tfrac{2}{7}\sqrt{7}\, R_1^{(1)} + \tfrac{5}{21}\sqrt{35}\, R_3^{(1)}. \\
R_1^{(1)} R_2^{(2)} &= \tfrac{5}{7}\sqrt{7}\, R_1^{(1)} + \tfrac{2}{21}\sqrt{35}\, R_3^{(1)} \\
&= \tfrac{1}{3}\sqrt{35}\, R_3^{(3)}. \\
R_1^{(1)} R_3^{(1)} &= \tfrac{5}{21}\sqrt{35}\, R_2^{(0)} + \tfrac{4}{33}\sqrt{55}\, R_4^{(0)} \\
&= \tfrac{2}{21}\sqrt{35}\, R_2^{(2)} + \tfrac{7}{33}\sqrt{55}\, R_4^{(2)}. \\
R_1^{(1)} R_3^{(3)} &= \tfrac{1}{3}\sqrt{35}\, R_2^{(2)} + \tfrac{2}{33}\sqrt{55}\, R_4^{(2)} \\
&= \tfrac{3}{11}\sqrt{55}\, R_4^{(4)}. \\
R_2^{(0)} R_2^{(0)} &= \sqrt{3}\, R_0^{(0)} - \tfrac{2}{9}\sqrt{7}\, R_2^{(0)} + \tfrac{50}{99}\sqrt{11}\, R_4^{(0)}. \\
R_2^{(0)} R_2^{(2)} &= \tfrac{4}{9}\sqrt{7}\, R_2^{(2)} + \tfrac{35}{99}\sqrt{11}\, R_4^{(2)}. \\
R_2^{(2)} R_2^{(2)} &= \sqrt{3}\, R_0^{(0)} + \tfrac{4}{9}\sqrt{7}\, R_2^{(0)} + \tfrac{8}{99}\sqrt{11}\, R_4^{(0)} \\
&= \tfrac{7}{9}\sqrt{7}\, R_2^{(2)} + \tfrac{14}{99}\sqrt{11}\, R_4^{(2)} \\
&= \tfrac{7}{11}\sqrt{11}\, R_4^{(4)}. \\
R_1^{(1)} R_4^{(0)} &= \tfrac{4}{33}\sqrt{55}\, R_3^{(1)} + \tfrac{7}{143}\sqrt{715}\, R_5^{(1)}. \\
R_1^{(1)} R_4^{(2)} &= \tfrac{7}{33}\sqrt{55}\, R_3^{(1)} + \tfrac{4}{143}\sqrt{715}\, R_5^{(1)} \\
&= \tfrac{2}{33}\sqrt{55}\, R_3^{(3)} + \tfrac{9}{143}\sqrt{715}\, R_5^{(3)}. \\
R_1^{(1)} R_4^{(4)} &= \tfrac{3}{11}\sqrt{55}\, R_3^{(3)} + \tfrac{2}{143}\sqrt{715}\, R_5^{(3)} \\
&= \tfrac{1}{13}\sqrt{715}\, R_5^{(5)}. \\
R_2^{(0)} R_3^{(1)} &= \tfrac{5}{21}\sqrt{35}\, R_1^{(1)} - \tfrac{8}{77}\sqrt{7}\, R_3^{(1)} + \tfrac{70}{429}\sqrt{91}\, R_5^{(1)}. \\
R_2^{(0)} R_3^{(3)} &= \tfrac{6}{11}\sqrt{7}\, R_3^{(3)} + \tfrac{15}{143}\sqrt{91}\, R_5^{(3)}. \\
R_2^{(2)} R_3^{(1)} &= \tfrac{2}{21}\sqrt{35}\, R_1^{(1)} + \tfrac{43}{77}\sqrt{7}\, R_3^{(1)} + \tfrac{28}{429}\sqrt{91}\, R_5^{(1)} \\
&= \tfrac{4}{11}\sqrt{7}\, R_3^{(3)} + \tfrac{21}{143}\sqrt{91}\, R_5^{(3)}. \\
R_2^{(2)} R_3^{(3)} &= \tfrac{1}{3}\sqrt{35}\, R_1^{(1)} + \tfrac{4}{11}\sqrt{7}\, R_3^{(1)} + \tfrac{8}{429}\sqrt{91}\, R_5^{(1)} \\
&= \tfrac{9}{11}\sqrt{7}\, R_3^{(3)} + \tfrac{6}{143}\sqrt{91}\, R_5^{(3)} \\
&= \tfrac{3}{13}\sqrt{91}\, R_5^{(5)}. \\
R_1^{(1)} R_5^{(1)} &= \tfrac{7}{143}\sqrt{715}\, R_4^{(0)} + \tfrac{2}{65}\sqrt{975}\, R_6^{(0)} \\
&= \tfrac{4}{143}\sqrt{715}\, R_4^{(2)} + \tfrac{3}{65}\sqrt{975}\, R_6^{(2)}. \\
R_1^{(1)} R_5^{(3)} &= \tfrac{9}{143}\sqrt{715}\, R_4^{(2)} + \tfrac{4}{195}\sqrt{975}\, R_6^{(2)} \\
&= \tfrac{2}{143}\sqrt{715}\, R_4^{(4)} + \tfrac{11}{195}\sqrt{975}\, R_6^{(4)}. \\
R_1^{(1)} R_5^{(5)} &= \tfrac{1}{13}\sqrt{715}\, R_4^{(4)} + \tfrac{2}{195}\sqrt{975}\, R_6^{(4)} \\
&= \tfrac{1}{15}\sqrt{975}\, R_6^{(6)}.
\end{aligned}$$

Since we are concerned with products of two Zernike functions, $R_{n_1}^{(l_1)} Y_{l_1}^{(m_1)} R_{n_2}^{(l_2)} Y_{l_2}^{(m_2)}$ that couple angular variables to products in the range $|l_1-l_2| \le l \le l_1+l_2$ via (57), the request is to expand $R_{n_1}^{(l_1)} R_{n_2}^{(l_2)}$ as $R_n^{(l)}$ with l fixed in that interval. The 3D analog to (24) and (26) is

$$R_{n_1}^{(l_1)}(r) R_{n_2}^{(l_2)}(r) \equiv \sum_{n_3=l_3}^{n_1+n_2} k_{n_1,l_1,n_2,l_2,n_3,l_3} R_{n_3}^{(l_3)}(r). \tag{58}$$

A k-sum rule is immediate, taking $r=1$ with (34). Triple insertion of (2) creates

$$k_{n_1,l_1,n_2,l_2,n_3,l_3} = \int_0^1 r^2 R_{n_1}^{(l_1)}(r) R_{n_2}^{(l_2)}(r) R_{n_3}^{(l_3)}(r) dr$$

$$= \sum_{s_1=0}^{\frac{n_1-l_1}{2}} \sum_{s_2=0}^{\frac{n_2-l_2}{2}} \sum_{s_3=0}^{\frac{n_3-l_3}{2}} \frac{1}{3+l_1+l_2+l_3+2(s_1+s_2+s_3)}$$

$$\times \prod_{t=1}^{3} \sqrt{2n_t+3}(-1)^{\frac{n_t-l_t}{2}+s_t} \binom{\frac{n_t-l_t}{2}}{s_t} \binom{\frac{1+n_t+l_t}{2}+s_t}{\frac{n_t-l_t}{2}}.$$

In practice, given the product of the radial polynomials as a polynomial in r, double insertion of (39) leads to the products of Table 17. Products of two entries of Tables 16 and 17, picking lines with matching l-parameters, expand products $Z_{n_1,l_1}^{(m_1)} Z_{n_2,l_2}^{(m_2)}$ of two 3D Zernike functions into a sum over 3D Zernike functions.

4. SUMMARY

The 2D and 3D Zernike functions are orthogonal basis sets defined in the unit circle and unit sphere. The specific notation introduced by Noll is the most common standard in 2D, and a more rational (but square-root loaded) standard seems to emerge in 3D. We have demonstrated the transformation of the basis functions between the radial-angular and the Cartesian systems for both dimensions.

The applications are numerical field simulations, where the foundation is backed by an isotropy in 2D or 3D space, but where followup calculations employ vector field operators which have simpler representations in global Cartesian than in local circular or spherical coordinates.

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ЗЕРНИКЕОВА БАЗА ЗА ТРАНСФОРМАЦИЈЕ ИЗ/У ДЕКАРТОВЕ КООРДИНАТЕ

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Стручни чланак

У раду су табеларно дати радијални полиноми 2D (кружних) и 3D (сферних) Зерникеових функција. Такође је дата обрнута зависност: радијална удаљеност на неки степен као коначни збир радијалних полинома, добијена на основу пројекција коришћењем особине ортогоналности полинома на јединичном интервалу.

Множењем полинома са угловном базом (азимут, поларни угао) дефинишу се Зерникеове функције за које су изведене трансформације из и у Декартове координате у систему са центром у средишту кружнице или сфере.