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SYMMETRIES IN CENTRAL-FORCE PROBLEMS

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SUMMARY: The two-body problem in central fields (reducible to a central-force problem) models a lot of concrete astronomical situations. The corresponding vector fields (in Cartesian and polar coordinates, extended via collision-blow-up and infinity-blow-up transformations) exhibit nice symmetries that form eight-element Abelian groups endowed with an idempotent structure. All these groups are isomorphic, which is not a trivial result, given the different structures of the corresponding phase spaces. Each of these groups contains seven four-element subgroups isomorphic to Klein's group. These symmetries are of much help in understanding various characteristics of the global flow of the general problem or of a concrete problem at hand, and are essential in searching for periodic orbits.

Key words. Celestial mechanics - Methods: analytical

1. INTRODUCTION

A lot of concrete situations, mainly to astronomy, can be tackled via the two-body problem associated to a central potential. Dynamics in classical fields, as those of Newton, Bertrand, Hall-Newcomb, Manev, Mücket-Treder, or in relativistic fields, as those of Schwarzschild, Einstein, Fock, Reissner-Nordström, Schwarzschild - de Sitter, etc., constitute an example in this sense. Dynamics in nongravitational fields: Coulomb, Van der Waals, a field of direct or re-emitted radiation, also joins this model. Considering a photogravitational field, whose gravitational component can be Newtonian or post-Newtonian (relativistic or not), we are within the same framework. The motion in the equatorial plane of a celestial body that generates a field featured only by zonal harmonics is modelled by the same problem. The model covers even less "astronomical" fields, as the elastic or gravito-elastic ones, or the force-free field. Such a framework provides a unifying point of view for a lot of problems of nonlinear particle dynamics. To have an idea about the two-body problem associated to (more or less concrete) central potentials, we quote arbitrarily Schwarzschild (1916), Wintner (1941), Fock (1959), Belenkii (1981), Chandrasekhar (1983), Cid et al. (1983), Damour and Schaefer (1987), Soffel (1989), Brumberg (1991), Blaga and Mioc (1992), Moeckel (1992), Ballinger and Diacu (1993), Mioc (1994, 2002, 2003), Delgado et al. (1996), Stoica and Mioc (1997), Mioc and Stavinschi (2001, 2002), and the references therein.

In this paper we tackle the two-body problem in an unspecified central field with a unique purpose: to point out the complex symmetries that characterize the corresponding vector field. After naturally reducing the problem to a central-force problem, we show that the equations of motion in both configuration-momentum coordinates and standard polar coordinates exhibit nice symmetries that form Abelian groups of order 8, endowed with an idempotent structure.

To go deeper into this investigation, we consider – as it is the case for the majority of the "astronomical" fields – that there exists the collisional singularity. To remove it, we resort to McGehee-type transformations (McGehee 1974), obtaining regularized equations of motion. These present the same symmetries, which form a group with the same characteristics.

Another limit situation is the escape, when the distance between the two particles tends to infinity. Treating this case also by McGehee-type transformations (McGehee 1973, 1974), we get groups with the same properties as above.

The groups of symmetries of the vector field in configuration-momentum or polar coordinates, in collision-blow-up coordinates, and in infinity-blow-up coordinates are isomorphic. This is not a trivial result, because the phase spaces associated to each coordinate system are not diffeomorphic, even if the corresponding McGehee-type transformations are real analytic diffeomorphisms. Each of these groups contains seven proper subgroups of order 4, isomorphic to Klein's group.

All these symmetries are particularly useful in understanding local flows and characteristics of the global flow, especially in cases that feature concrete astronomical situations. Moreover, they are essential in the search for periodic orbits in most problems of celestial mechanics.

2. BASIC EQUATIONS

Let the nature of the field be unspecified; we know only that it is central. So, the associated two-body problem can be reduced to a central-force problem, and the relative motion is confined to a plane. The equations of motion are featured by the Hamiltonian $H(\mathbf{q},\mathbf{p}) = |\mathbf{p}|^2/2 - U(\mathbf{q})$, where $\mathbf{q} = (q_1, q_2) \in \mathbf{R}^2\backslash\{(0,0)\}$ and $\mathbf{p} = (p_1, p_2) \in \mathbf{R}^2$ are the position (configuration) vector and the momentum vector, respectively, whereas $U(\mathbf{q})$ is the unspecified potential function. Characterizing the field by the potential U, we used the well-known result that says: every central field is potential, and its potential energy depends only on the distance to the field centre: $-U(\mathbf{q}) = -U(|\mathbf{q}|)$. The problem clearly admits the first integrals of energy ($H(\mathbf{q},\mathbf{p}) = h =$ constant) and angular momentum ($L(\mathbf{q},\mathbf{p}) = q_1p_2 - q_2p_1 = C =$ constant).

Written in scalar form, the equations of motion explicitly read

$$\begin{aligned} \dot{q}_1 &= p_1, \quad \dot{q}_2 = p_2, \\ \dot{p}_1 &= [\partial U(|\mathbf{q}|)/\partial|\mathbf{q}|](q_1/|\mathbf{q}|), \\ \dot{p}_2 &= [\partial U(|\mathbf{q}|)/\partial|\mathbf{q}|](q_2/|\mathbf{q}|) \end{aligned} \tag{1}$$

3. SYMMETRIES IN CARTESIAN AND POLAR CORDINATES

As it is easy to verify, the vector field (1) exhibits seven nice symmetries that map solution onto solution. These symmetries, $S_i = S_i(q_1, q_2, p_1, p_2, t)$, $i = \overline{1,7}$, are

$$\begin{aligned} S_1 &= (q_1, q_2, -p_1, -p_2, -t), \\ S_2 &= (q_1, -q_2, p_1, -p_2, t), \\ S_3 &= (-q_1, q_2, -p_1, p_2, t), \\ S_4 &= (q_1, -q_2, -p_1, p_2, -t), \\ S_5 &= (-q_1, q_2, p_1, -p_2, -t), \\ S_6 &= (-q_1, -q_2, -p_1, -p_2, t), \\ S_7 &= (-q_1, -q_2, p_1, p_2, -t). \end{aligned} \tag{2}$$

Among these symmetries, only three are mutually independent. Indeed, let these ones be S_1, S_2, S_3. One can immediately check that $S_4 = S_1 \circ S_2$, $S_5 = S_1 \circ S_3$, $S_6 = S_2 \circ S_3$, $S_7 = S_1 \circ S_2 \circ S_3$. A similar structure is recovered for any three symmetries considered as independent of each other.

The set $G = \{I\} \cup \{S_i \mid i = \overline{1,7}\}$ (where I is the identity), endowed with the composition law "$\circ$", forms a symmetric Abelian group. To prove this, the composition table below, easy to construct and check, is sufficient.

$\circ$	I	S_1	S_2	S_3	S_4	S_5	S_6	S_7
I	I	S_1	S_2	S_3	S_4	S_5	S_6	S_7
S_1	S_1	I	S_4	S_5	S_2	S_3	S_7	S_6
S_2	S_2	S_4	I	S_6	S_1	S_7	S_3	S_5
S_3	S_3	S_5	S_6	I	S_7	S_1	S_2	S_4
S_4	S_4	S_2	S_1	S_7	I	S_6	S_5	S_3
S_5	S_5	S_3	S_7	S_1	S_6	I	S_4	S_2
S_6	S_6	S_7	S_3	S_2	S_5	S_4	I	S_1
S_7	S_7	S_6	S_5	S_4	S_3	S_2	S_1	I

Also observe that every element is its own inverse, hence G has an idempotent structure.

Now, let us pass to standard polar coordinates via the real analytic diffeomorphisms

$$\begin{aligned} r &= |\mathbf{q}|, \\ \theta &= \arctan(q_2/q_1), \\ u &= \dot{r} = (q_1p_1 + q_2p_2)/|\mathbf{q}|, \\ v &= r\dot{\theta} = (q_1p_2 - q_2p_1)/|\mathbf{q}|, \end{aligned} \tag{3}$$

which make the equations of motion become

$$\begin{aligned} \dot{r} &= u, \\ \dot{\theta} &= v/r, \\ \dot{u} &= v^2/r + \partial U(r)/\partial r, \\ \dot{v} &= -uv/r. \end{aligned} \tag{4}$$

The energy and angular momentum integrals read $u^2 + v^2 = 2[U(r) + h]$ and $rv = C$, respectively.

The vector field (4) also has seven nice symmetries, $S_i^{pol} = S_i^{pol}(r,\theta,u,v,t)$, $i = \overline{1,7}$, as follows

$$\begin{aligned} S_1^{pol} &= (r,\theta,-u,-v,-t),\\ S_2^{pol} &= (r,-\theta,u,-v,t),\\ S_3^{pol} &= (r,\pi-\theta,u,-v,t),\\ S_4^{pol} &= (r,-\theta,-u,v,-t),\\ S_5^{pol} &= (r,\pi-\theta,-u,v,-t),\\ S_6^{pol} &= (r,\pi+\theta,u,v,t),\\ S_7^{pol} &= (r,\pi+\theta,-u,-v,-t). \end{aligned} \tag{5}$$

It is easy to verify that equations (4) are invariant to these transformations. It is also easy to see that only three symmetries S_i^{pol}, $i = \overline{1,7}$, are mutually independent, and that the set $G^{pol} = \{I\} \cup \{S_i^{pol} \mid i = \overline{1,7}\}$, endowed with the same composition law "$\circ$" as G, forms a symmetric Abelian group with an idempotent structure.

Considering the real analytic diffeomorphism $(\mathbf{R}^2\backslash\{(0,0)\}) \times \mathbf{R}^3 \to (0,+\infty) \times S^1 \times \mathbf{R}^3$, $(q_1,q_2,p_1,p_2,t) \mapsto (r,\theta,u,v,t)$, it is clear that G and G^{pol} are diffeomorphic.

Let us see what symmetries (5) physically signify. Considering separately the transformations for each variable, $(t,-t)$ corresponds to motion in the future/past; $(u,-u)$ means outwards/inwards motion; $(v,-v)$ means clockwise/counterclockwise motion; finally, $(\theta,-\theta)$, $(\theta,\pi-\theta)$, $(\theta,\pi+\theta)$ correspond to positions shifted with respect to each other by 2θ, $\pi-2\theta$, and π, respectively. As to their combination into symmetries, S_1^{pol} corresponds to the reversibility of the flow: for every solution there is another solution that has the same coordinates and opposite velocities, all in reversed time. S_2^{pol} implies the fact that, for every solution, there is another solution with opposite θ and v coordinates, and so forth.

4. SYMMETRIES IN COLLISION-BLOW-UP AND INFINITY-BLOW-UP COORDINATES

In most of the concrete cases met in physics and astronomy, the potential $U(r)$ has an isolated singularity at the origin $r = 0$. In all such cases, this singularity (of both equations of motion and solutions) corresponds to a collision (see, e.g., Wintner 1941, Mioc and Stavinschi 2001, 2002).

It is clear that the collision singularity does exist only for $\lim_{r\to 0} U(r) = +\infty$. Indeed, the situation $\lim_{r\to 0} U(r) = -\infty$ is contradicted by the energy integral. The former situation means that there exists a value $r_1 > 0$ for which $U(r) > 0$, $\forall r < r_1$. To study collisions, we shall suppose $r < r_1$, and apply the McGehee-type transformations of the second kind (McGehee 1974):

$$\begin{aligned} x &= u/\sqrt{U(r)}, \quad y = v/\sqrt{U(r)};\\ ds &= [\sqrt{U(r)}/r]dt. \end{aligned} \tag{6}$$

Under these real analytic diffeomorphisms, the vector field (4) becomes

$$\begin{aligned} r' &= rx,\\ \theta' &= y,\\ x' &= (1-x^2/2)[r/U(r)][\partial U(r)/\partial r] + y^2,\\ y' &= -(xy/2)\{[r/U(r)][\partial U(r)/\partial r] + 2\}, \end{aligned} \tag{7}$$

where $(')=\mathrm{d}/\mathrm{d}s$, and we kept, by abuse, the same notation for the functions of the new timelike variable s. The integrals of energy and angular momentum become $x^2+y^2 = 2[1+h/U(r)]$ and $\sqrt{U(r)}ry = C$, respectively.

Let us now formulate an extra-hypothesis, verified in most concrete physical and astronomical situations: $\lim_{r\to 0}\{[r/U(r)][\partial U(r)/\partial r]\}$ exists and is finite. In this case, the vector field (7) is regular and the phase space extends smoothly to the boundary $r = 0$ (which becomes the collision manifold).

This extended vector field benefits of the same symmetries as (4). To prove this, consider the real analytic diffeomorphism $(0,+\infty) \times S^1 \times \mathbf{R}^3 \to [0,+\infty) \times S^1 \times \mathbf{R}^3$, $(r,\theta,u,v,t) \mapsto (r,\theta,x,y,s)$, given by (6), and define $\tilde{S}_i(r,\theta,x,y,s) = S_i^{pol}(r,\theta,u,v,t)$, $i = \overline{1,7}$. By (5), the invariance of (7) to the transformations $\tilde{S}_i$ results immediately.

As in the previous cases, only three symmetries $\tilde{S}_i$ are mutually independent. The set $G_0 = \{I\} \cup \{\tilde{S}_i \mid i = \overline{1,7}\}$, endowed with the same composition law as G^{pol}, also forms a symmetric Abelian group with an idempotent structure.

To consider another limit situation, there are cases (as the Schwarzschild - de Sitter field; e.g., Blaga and Mioc 1992) in which the potential tends to infinity for $r \to +\infty$ (escape/capture). In such a case, one brings the infinity at the origin via the McGehee-type transformation of the first kind (McGehee 1973) $\rho = 1/r$, which makes (7) to become

$$\begin{aligned} \rho' &= -\rho x,\\ \theta' &= y,\\ x' &= (x^2/2-1)[\rho/U(\rho)][\partial U(\rho)/\partial\rho] + y^2,\\ y' &= (xy/2)\{[\rho/U(\rho)][\partial U(\rho)/\partial\rho] - 2\}. \end{aligned} \tag{8}$$

The integral of energy now reads $x^2+y^2 = 2[1+h/U(\rho)]$. Considering the supplementary hypothesis that $\lim_{\rho\to 0}\{[\rho/U(\rho)][\partial U(\rho)/\partial\rho]\}$ exists and is finite, the vector field (8) is regular and the phase space extends smoothly to the boundary $\rho = 0$ (which becomes the infinity manifold).

Considering the real analytic diffeomorphism $(r,\theta,x,y,s) \mapsto (\rho,\theta,x,y,s)$, we are in the position to define $\hat{S}_i(\rho,\theta,x,y,s) = \tilde{S}_i(r,\theta,x,y,s)$, $i = \overline{1,7}$.

The invariance of (8) to $\hat{S}_i$ results immediately. This means that the vector field (8) has the same symmetries as (7).

As expected, only three symmetries $\hat{S}_i$ are mutually independent. The set $G_\infty = \{I\} \cup \{\hat{S}_i \mid i = \overline{1,7}\}$, endowed with the same composition law "$\circ$", also forms a symmetric Abelian group with an idempotent structure.

But the most frequent situation (encountered especially in astronomy) is the one for which $r \to +\infty$ makes the potential tend to zero (dynamics in Newton's field is perhaps the most suggestive example). In such a case, it is clear that the groups G or G^{pol} persist. Alternatively, one can apply suitable McGehee-type transformations to (8), getting a new group of symmetries (G'_∞, say) with exactly the same properties as G, G^{pol}, G_0, and G_∞.

5. MAIN RESULTS

Theorem 1. *The equations of motion of a central-force problem benefit of symmetries that form Abelian eight-element groups endowed with an idempotent structure. Such groups persist in either Hamiltonian* (G) *and polar* (G^{pol}) *coordinates, or collision-blow-up* (G_0) *and infinity-blow-up* (G_∞, G'_∞) *coordinates.*

Proof. See Sections 3 and 4.

Theorem 2. *The groups* G, G^{pol}, G_0, G_∞ *and* G'_∞ *, are isomorphic.*

Proof. Since all these groups are commutative of order 8, with three generators of order 2, by the Fundamental Theorem of Abelian Groups, they are all isomorphic to $\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$... The result is proved.

Remark 3. Theorem 2 is not a trivial result. The phase spaces associated to each coordinate system are not diffeomorphic. Compared to the phase spaces corresponding to G and G^{pol} (diffeomorphic each other), the one corresponding to G_0 contains the supplementary (boundary) collision manifold. Within a similar comparison, the phase spaces corresponding to G_∞ or G'_∞ (diffeomorphic each other) contain the supplementary (boundary) infinity manifold.

Theorem 4. *Each of the above groups contains seven proper subgroups of order 4, isomorphic to Klein's group.*

Proof. Consider first the group G. Consider three elements in $G: S_i, S_j, S_k$ ($\neq I$), such that $S_i \circ S_j = S_k$, $i \neq j \neq k \neq i$. Composing successively both members of this equality with S_i and S_k, and taking into account the properties of G, we first get $S_i \circ S_k = S_j$, then $S_j \circ S_k = S_i$. It is clear that the set $H_{ijk} = \{I\} \cup \{S_i, S_j, S_k\}$ is a proper subgroup of G. Checking the composition table of G, we find that there are exactly seven such subgroups, namely $H_{124} = \{I, S_1, S_2, S_4\}$ and, analogously, H_{135}, H_{167}, H_{236}, H_{257}, H_{347}, H_{456}. Each one is Abelian of order 4, with two generators of order 2, and with an idempotent structure, hence isomorphic to Klein's group.

Let us define, in the same manner, the subgroups H^{pol}_{ijk}, $\tilde{H}_{ijk}$, $\hat{H}_{ijk}$, H'_{ijk} of the groups G^{pol}, G_0, G_∞, G'_∞, respectively. It is easy to check that, for each group (G), there are exactly seven subgroups (H), isomorphic to Klein's group. This completes the proof.

Remark 5. Many of these subgroups correspond to clear physical situations. For instance, H_{124}, H_{135}, H_{257}, H_{347} correspond to symmetries in which q_1, q_2, p_1, p_2 keep their sign, respectively. In H_{236}, t keeps its sign, and it is the same for H^{pol}_{236} (where, according to the first formula (4), u keeps its sign, too). In H^{pol}_{456}, v keeps its sign, and so forth.

The symmetries revealed out in this paper are of much help in understanding various characteristics of the global flow of either the very general problem or of a concrete problem at hand. Indeed, for each solution proved to exist, they show the existence of many other solutions.

To give some examples, consider the two-body problems associated to some concrete fields: Manev (Delgado et al. 1996, Diacu et al. 2000), Schwarzschild (Schwarzschild 1916, Stoica and Mioc 1997), or Fock (Fock 1959, Mioc 1994). Given an orbit of the type ejection-escape, ejection - unstable equilibrium, unstable equilibrium - escape, etc., the reciprocal orbits of the type capture-collision, unstable equilibrium - collision, capture - unstable equilibrium, etc., do certainly exist via the symmetries proved in this paper. It is the same for the periodic or quasiperiodic orbits performed in one sense or in another.

Even if this is not a central-force case, to emphasize the importance of symmetries, we shall give another example. This is based on the sixteen heteroclinic orbits met in the collision and/or infinity manifold of the two-body problem associated to the anisotropic Kepler (Gutzwiller 1973), Manev (Craig et al. 1999), or Schwarzschild (Mioc et al. 2003) models. It was proved that the existence of only two such orbits is sufficient to find – via the same symmetries – all remaining orbits.

Moreover, these symmetries are very useful to find symmetric periodic orbits – essentially by means of the continuation method – in perturbed two-body problems depending on a small parameter ε, such that, as usual, by $\varepsilon = 0$ we recover the unperturbed problem. This especially helps to the study of the restricted three-body problem, either in post-Newtonian fields, or in the case when one of the primaries is an oblate planet that re-emits radiation, or is a rotating star. As Diacu (2003) showed, symmetries play an essential role in searching for periodic orbits in most of the problems of celestial mechanics. Of course, a lot of non-astronomical restricted three-body problems can also be considered.

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СИМЕТРИЈЕ У ПРОБЛЕМИМА ЦЕНТРАЛНИХ СИЛА

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Оригинални научни рад

Проблем два тела за централне силе (своднлив на проблем централног кретања честице) даје модел за многе астрономске ситуације. Одговарајућа векторска поља (у Декартовим и поларним координатама, рађена преко трансформација одстрањивањем колизије и бесконачности) показују лепе симетрије које образују осмочлане абеловске групе са идемпотентном структуром. Све ове групе су изоморфне што није тривијалан резултат када су дате различите структуре одговарајућих фазних простора. Свака од ових група садржи седам подгрупа од по четири члана изоморфне са Клајновом групом. Ове симетрије су од велике помоћи у разумевању разних карактеристика глобалног дотока општег проблема или конкретног проблема који се решава и, осим тога, су од суштинског значаја у трагању за периодичним орбитама.