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GENERALIZED SCHUSTER LAW AND KING'S FORMULA

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SUMMARY: By varying a particular form of the generalised Schuster density law (the exponent in the denominator equal to $\frac{3}{2}$) the authors look for a suitable substitution for King's density formula usually applied to star clusters and dwarf galaxies. The authors find expressions yielding almost identical density values as King's formula, but from the mathematical point of view significantly more simple for use.

1. INTRODUCTION

As well known, King (1962) proposed a formula describing the mass distribution in stellar systems like open clusters, globular clusters and dwarf galaxies. Since that time this formula has been amply applied for the purpose of fitting the star counts in these systems (e. g. Meylan, 1988). Though these fits can be considered as successful, the application of King's formula presents a serious difficulty since it permits no analytical solution of the Poisson equation. On the other hand, in some applications, for example the orbit calculation, it is desirable to have an analytical expression at least for the gravitation-field strength. Therefore, the decision of the present authors is to look for a more simple density formula yielding an almost equally qualitative fit to the observations. In our opinion the generalized Schuster density law (e. g. Lohmann, 1964) offers such a possibility, especially bearing in mind that recently one of us (Ninković, 1998) demonstrated that King's density law appears as an asymptotical case of the generalized Schuster density law with $\beta = \frac{3}{2}$ (β the exponent in the denominator); consequently both yield an infinite total mass if integrated to infinity. Like King's formula this special case of the generalised Schuster density law has also three parameters so that any comparison (local or global) for equal central density and characteristic radius (called also core radius) becomes simple.

2. FORMALISM

The particular form of the generalized Schuster density law mentioned in the preceding Section is

$$\rho(r) = \frac{\rho(0)}{[1 + (r/r_c)^2]^{3/2}} \ , \ r \leq r_l \ ;$$

$$\rho(r) = 0 \ , \ r > r_l \ , \ r_c = const \ .$$

This expression is in Anglo-Saxon literature usually referred to as the modified Hubble-Reynolds formula (e. g. Binney and Tremaine, 1987 - p. 39). Since in King's formula the density vanishes at the boundary ($r = r_l$, r_l limiting radius, i. e. tidal radius in King's original terminology), our first modification will be to introduce an additional term producing the same effect in our case. Besides, we shall treat King's formula as if it represented exactly the mass distribution in the types of stellar systems mentioned above. Therefore, for the purpose of achieving a fit as good as possible we introduce other modifications to have finally

$$\rho = K\Big(1 - \frac{x^2}{x_l^2}\Big)^i\Big[\frac{1}{(1+x^2)^{3/2}} - \frac{1}{(1+x_l^2)^{3/2}}\Big] , \quad (1)$$

where K is a constant (obviously, it has the same dimension as the density), x is the dimensionless radius - $x = r/r_c$, also $x_l = r_l/r_c$ - whereas i is a non negative integer. With regard to the purpose of the present paper the values of interest are $i = 0$, $i = 1$ and $i = 2$. The case $i = 0$ is the most simple and it corresponds to the first modification mentioned above since the second term in the brackets is what enables the density to reach zero at the boundary. By using (1) it is possible to obtain analytically the mass within a given radius and also the potential, but the integral yielding the total potential energy of the system is not obtainable analytically (e. g. Ninković, 1994). As for the potential we use the following general expression

$$\Pi(r) = \frac{G\mathcal{M}(r)}{r} + 4\pi G\int_r^{r_l} \rho(r) r\, dr \ ;$$

here G is the gravitation constant and $\mathcal{M}(r)$ is the mass inside a given radius r. The corresponding expressions are given in Appendix.

3. RESULTS

As already said in the Introduction, King's formula and (1) can be compared locally and globally. In the former case the comparison is performed through the density amounts. As for the latter one, one compares the resulting total masses. In order to make such comparisons more clear we shall rewrite King's density formula followed by a corresponding total-mass expression. The volume-density expression as given by King (formulae (27)-(29) of his paper) is

$$\rho = \frac{k}{\pi r_c(1+x_l^2)^{3/2}} \quad \frac{1}{z^2}\Big[\frac{1}{z}\arccos z - (1-z^2)^{1/2}\Big] ,$$

$$z = \Big[\frac{1+x^2}{1+x_l^2}\Big]^{1/2} .$$

k is a constant with surface-density dimensions. It can be easily substituted by the central density by means of the following expression

$$k = \pi r_c \rho(0)\Big[\arccos\frac{1}{(1+x_l^2)^{1/2}} - \frac{x_l}{1+x_l^2}\Big]^{-1} . \quad (2)$$

As already said, the integration of the density given by this formula yields no analytical expression for the mass inside an arbitrary radius, i. e. total mass. However, the total mass can be calculated by using the corresponding surface density. The amount obtained in this way is

$$\mathcal{M} = \pi k r_c^2 f(x_l) \ ,$$

where for k one should substitute expression (2), whereas $f(x_l)$ is a (dimensionless) function of x_l given by

$$f(x_l) = \ln q + \frac{4}{q^{1/2}} + \frac{x_l^2}{q} - 4 \ ,$$

$$q = 1 + x_l^2 \ .$$

Now one can carry out the envisaged comparisons. Both distributions are characterized by three parameters: $\rho(0)$, r_c and x_l. Therefore, any comparison is meaningful if these three parameters have the same values. In the local comparisons r_c will be used as distance unit, and $\rho(0)$ as density unit respectively. In the global comparisons the mass will be expressed in the units of $\rho(0)r_c^3$. According to the existing evidence values less than 1 for x_l are meaningless; mass distributions within open clusters (e. g. King, 1962) and dwarf galaxies (e. g. Lake,1990; Pryor and Kormendy, 1990) are fitted with $x_l \sim 10^0$, whereas in the case of globular ones the corresponding order of magnitude is 10^1 (e. g. Kukarkin and Kireeva, 1979). Therefore, the present comparisons will have a lower limit $x_l = 1$.

The least agreement is achieved in the case $i = 0$. In local comparisons the fit is very good at $x \leq 1$, especially for high x_l (say $x_l \geq 50$). However, with higher x it becomes worse. In general the best illustration of the agreement is realised through the global comparison. Formula (1) - $i = 0$ - yields systematically a higher total mass than King's formula. The highest mass ratio is with $x_l = 1$ (1.47), as x_l increases the mass ratio decreases to attain about 1.16 at $x_l = 1000$. Of course, as already said above, when x_l tends to infinity the mass ratio tends to 1 (also valid for any other i). The dependence of the mass ratio on x_l is presented in Fig. 1.

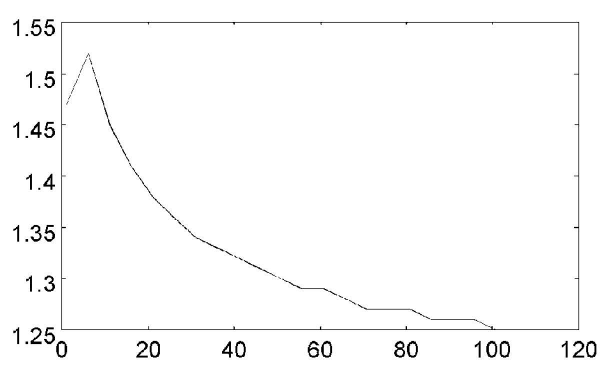

Fig. 1. *The ratio of the total masses - distribution (1), $i = 0$, according to King's formula - as function of x_l; the mass unit is $\rho(0)r_c^3$.*

The case $i = 1$ yields a better agreement. Differences in the density values are very often of the order of 10^{-5}. As for the global agreement, the mass ratio is about 0.91 at $x_l = 1$, to attain about 1.23 at $x_l = 10$. Afterwards it gradually decreases (for example if x_l is 1000 it is about 1.1). This agreement is presented in Fig. 2.

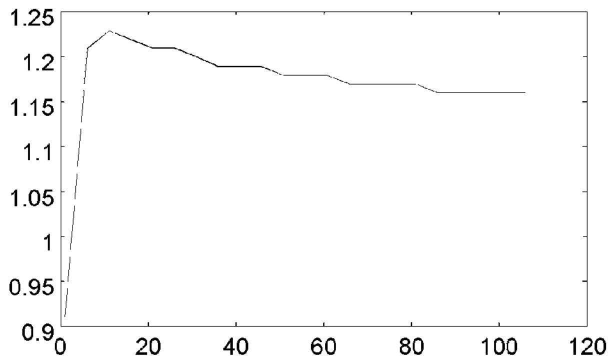

Fig. 2. *The ratio of the total masses - distribution (1), $i = 1$, according to King's formula - as function of x_l; the mass unit is $\rho(0)r_c^3$.*

The last case $i = 2$ yields also a good agreement. This time the total mass is systematically lower than that emanating from King's formula. The mass ratio is especially low at $x_l = 1$ - about 0.2; it increases for higher x_l values (for example about 0.73 at $x_l = 10$) to become close to one at x_l sufficiently high (say at $x_l = 50$ it is about 0.95, at $x_l = 100$ about 0.97, at $x_l = 1000$ about 0.99, etc); Fig. 3 presents the agreement. As for the local comparisons, the differences in the density values are also very often of the order of 10^{-5}.

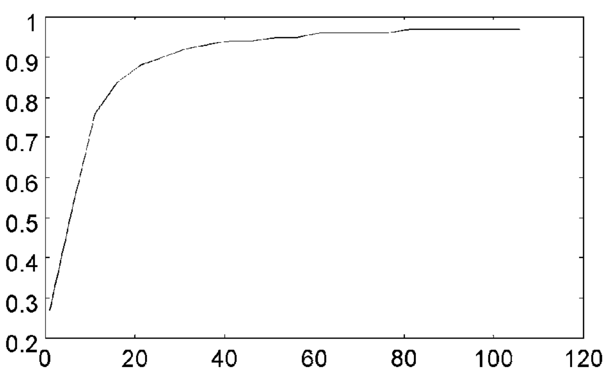

Fig. 3. *The ratio of the total masses - distribution (1), $i = 2$, according to King's formula - as function of x_l; the mass unit is $\rho(0)r_c^3$.*

4. DISCUSSION AND CONCLUSIONS

It is established that the mass distribution expressed by means of (1) fits well that resulting from King's formula. As a general conclusion one may say that out of the three special cases of (1) examined here two of them - $i = 1$ and $i = 2$ - yield especially good agreements with King's formula. In particular, the former case yields an especially good agreement for about $x_l \leq 10$, whereas the latter one yields an especially good agreement for higher values of x_l. In view of what has been said above concerning the application of King's formula to concrete stellar systems the case $i = 1$ (formula (1)) could be suitable for application to open clusters and dwarf galaxies, whereas the other one could be applicable to globular clusters.

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APPENDIX

I Mass within a given radius

i) case i=0

$$\mathcal{M}(x) = 4\pi K r_c^3 I \ ,$$

$$I = I_1 + I_2 \ ,$$

$$I_1 = I_0 - \sin\varphi \ , \ I_0 = \ln|\tan(\frac{\varphi}{2} + \frac{\pi}{4})| \ , \ \varphi = \arctan x$$

$$I_2 = -\frac{1}{(1 + x_l^2)^{3/2}} \frac{x^3}{3} \ .$$

ii) case i=1

$$\mathcal{M}(x) = 4\pi K r_c^3 I \ ,$$

$$I = I_1 + I_2 + I_3 + I_4 \ ,$$

$$I_1 = I_0 - \sin\varphi \ , \ I_0 = \ln|\tan(\frac{\varphi}{2} + \frac{\pi}{4})| \ , \ \varphi = \arctan x$$

$$I_2 = -\frac{1}{(1 + x_l^2)^{3/2}} \frac{x^3}{3} \ ,$$

$$I_3 = -\frac{1}{x_l^2}\Big(\frac{\sin\varphi}{2\cos^2\varphi} - \frac{3}{2}I_0 + \sin\varphi\Big) \ ,$$

$$I_4 = \frac{1}{(1+x_l^2)^{3/2}}\frac{1}{x_l^2}\frac{x^5}{5} \ .$$

iii) case i=2

$$\mathcal{M}(x) = 4\pi K r_c^3 I \ ,$$

$$I = I_1 + I_2 + I_3 + I_4 + I_5 + I_6 \ ,$$

$$I_1 = I_0 - \sin\varphi \ , \ I_0 = \ln|\tan(\frac{\varphi}{2} + \frac{\pi}{4})| \ , \ \varphi = \arctan x$$

$$I_2 = -\frac{1}{(1+x_l^2)^{3/2}}\frac{x^3}{3} \ ,$$

$$I_3 = -\frac{2}{x_l^2}\Big(\frac{\sin\varphi}{2\cos^2\varphi} - \frac{3}{2}I_0 + \sin\varphi\Big) \ ,$$

$$I_4 = \frac{2}{(1+x_l^2)^{3/2}}\frac{1}{x_l^2}\frac{x^5}{5} \ ,$$

$$I_5 = \frac{1}{x_l^4}\Big(\frac{\sin\varphi}{4\cos^4\varphi} - \frac{9}{8}\frac{\sin\varphi}{\cos^2\varphi} + \frac{15}{8}I_0 - \sin\varphi\Big) \ ,$$

$$I_6 = -\frac{1}{(1+x_l^2)^{3/2}}\frac{1}{x_l^4}\frac{x^7}{7} \ .$$

II Potential

i) case i=0

$$\Pi(x) = \frac{G\mathcal{M}(x)}{r_c x} + 4\pi G K r_c^2 J \ ,$$

$$J = J_1 + J_2 \ ,$$

$$J_1 = \cos\varphi - \cos\varphi_0 \ , \ \varphi = \arctan x \ , \ \varphi_0 = \arctan x_l \ ,$$

$$J_2 = -\frac{1}{2}\frac{1}{(1+x_l^2)^{3/2}}(x_l^2 - x^2) \ .$$

ii) case i=1

$$\Pi(x) = \frac{G\mathcal{M}(x)}{r_c x} + 4\pi G K r_c^2 J \ ,$$

$$J = J_1 + J_2 + J_3 + J_4 \ ,$$

$$J_1 = \cos\varphi - \cos\varphi_0 \ , \ \varphi = \arctan x \ , \ \varphi_0 = \arctan x_l \ ,$$

$$J_2 = -\frac{1}{2}\frac{1}{(1+x_l^2)^{3/2}}(x_l^2 - x^2) \ ,$$

$$J_3 = -\frac{1}{x_l^2}\Big[\Big(\frac{1}{\cos\varphi_0} - \frac{1}{\cos\varphi}\Big) - (\cos\varphi - \cos\varphi_0)\Big] \ ,$$

$$J_4 = \frac{1}{4}\frac{1}{x_l^2}\frac{1}{(1+x_l^2)^{3/2}}(x_l^4 - x^4) \ .$$

iii) case i=2

$$\Pi(x) = \frac{G\mathcal{M}(x)}{r_c x} + 4\pi G K r_c^2 J \ ,$$

$$J = J_1 + J_2 + J_3 + J_4 + J_5 + J_6 \ ,$$

$$J_1 = \cos\varphi - \cos\varphi_0 \ , \ \varphi = \arctan x \ , \ \varphi_0 = \arctan x_l \ ,$$

$$J_2 = -\frac{1}{2}\frac{1}{(1+x_l^2)^{3/2}}(x_l^2 - x^2) \ ,$$

$$J_3 = -\frac{2}{x_l^2}\Big[\Big(\frac{1}{\cos\varphi_0} - \frac{1}{\cos\varphi}\Big) - (\cos\varphi - \cos\varphi_0)\Big] \ ,$$

$$J_4 = \frac{2}{4}\frac{1}{x_l^2}\frac{1}{(1+x_l^2)^{3/2}}(x_l^4 - x^4) \ .$$

$$J_5 = \frac{1}{x_l^4}\Big[\frac{1}{3}\Big(\frac{1}{\cos^3\varphi_0} - \frac{1}{\cos^3\varphi}\Big) - 2\Big(\frac{1}{\cos\varphi_0} - \frac{1}{\cos\varphi}\Big) + (\cos\varphi - \cos\varphi_0)\Big] \ ,$$

$$J_6 = -\frac{1}{6}\frac{1}{x_l^4}\frac{1}{(1+x_l^2)^{3/2}}(x_l^6 - x^6) \ .$$

УОПШТЕНИ ШУСТЕРОВ ЗАКОН И КИНГОВА ФОРМУЛА

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Оригинални научни рад

Варирањем једног конкретног облика уопштеног Шустеровог закона (изложилац у имениоцу износи $\frac{3}{2}$) аутори траже погодну замену за Кингову формулу за густину која се обично користи код звезданих јата и патуљастих галаксија. Аутори проналазе изразе који дају скоро исте вредности густине као и Кингова формула, али су са математичке тачке гледишта знатно једноставнији за рад.